CHAPTER 1

PRELIMINARIES

1.1 Real Numbers and the Real Line

Calculus is based on the real number system. Real numbers are numbers that can be expressed as decimals.
We distinguish three special subsets of real numbers:

1. The **natural numbers**, namely 1, 2, 3, 4,…
2. The **integers**, namely 0, ±1, ±2, ±3,…
3. The **rational numbers**, which are ratios of integers. These numbers can be expressed in the form of a function m/n, where m and n are integers and $n \neq 0$. Examples are:

$$\frac{1}{2},\ -\frac{5}{3} = \frac{-5}{3} = \frac{5}{-3},\ \frac{200}{13},\ 67 = \frac{67}{1}$$

(Recall that division by is always ruled out, so expressions like $\frac{3}{0}$ and $\frac{0}{0}$ are undefined.)

The real numbers can be represented geometrically as points on a number line called the **real line,** as in Figure 1.1.

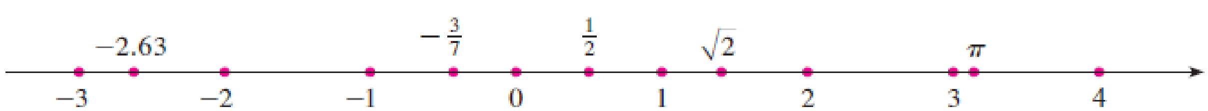


Figure 1.1

1.1.1 Intervals

Certain sets (or a subset) of real numbers, called intervals, occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from to consists of all numbers between a and b is denoted by the symbol (a, b). Using set-builder notation, we can write:

$$(a, b) = \{x|\ a < x < b\}$$

(which is read "(a, b) is the set of x such that x is an integer and $a < x < b$.)

Notice that the endpoints of the interval -namely, a and b- are excluded. This is indicated by the round brackets and by the open dots in Table 1.1. The **closed interval** from a to b is the set

$$[a, b] = \{x|\ a \le x \le b\}$$

Here the endpoints of the interval are included. This is indicated by the square brackets [ ] and by the solid dots in table 1.1. It is also possible to include only one endpoint in an interval, as shown in Table 1.1.

Table 1.1

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

1.1.2 Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

The following useful rules can be derived from them, where the symbol => means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

Example 1: Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x - 1} \geq 5$

Solution:

(a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \quad \text{Add 1 to both sides.}$$

$$x < 4 \quad \text{Subtract } x \text{ from both sides.}$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).

(b)

$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \quad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \quad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \quad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \quad \text{Divide by 7.}$$

The solution set is the open interval $(-3/7, \infty)$ (Figure 1.1b).

The inequality $6/(x - 1) \geq 5$ can hold only if $x > 1$ because otherwise $6/(x - 1)$ is undefined or negative. Therefore, $(x - 1)$ is positive and the inequality will be preserved if we multiply both sides by $(x - 1)$ and we have

$$\frac{6}{x-1} \geq 5$$

$$6 \geq 5x - 5 \qquad \text{Multiply both sides by } (x-1).$$

$$11 \geq 5x \qquad \text{Add 5 to both sides.}$$

$$\frac{11}{5} \geq x. \qquad \text{Or } x \leq \frac{11}{5}.$$

The solution set is the half-open interval (1, 11/5] (Figure 1.1c)

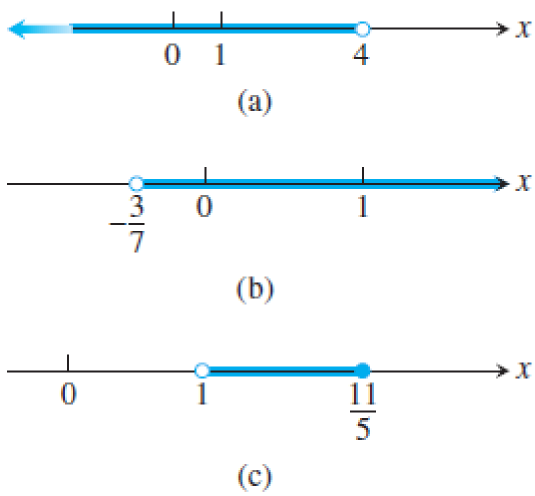


Figure 1.2

1.1.3 Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is the distance from x to 0 on the real number line. Distances are always positive or 0, so we have

$$|x| \geq 0 \quad \text{for every number } x$$

Or it can be defined by the formula:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Example 2:

$|3| = 3$, $|0| = 0$, $|-5| = -(-5) = 5$, $|-|a|| = |a|$

Geometrically, the absolute value of x is the distance from x to 0 on the real number line. Since distances are always positive or 0, we see that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$. Also,

$$|x - y| = \text{the distance between } x \text{ and } y$$

on the real line (Figure 1.2).

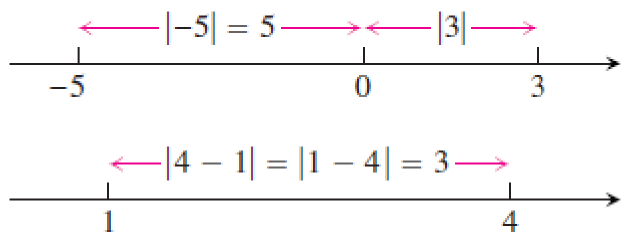


Figure 1.3

The absolute value has the following properties:

Absolute Value Properties

1.	$\lvert -a \rvert = \lvert a \rvert$	A number and its additive inverse or negative have the same absolute value.
2.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.
3.	$\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a \rvert}{\lvert b \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4.	$\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Example 3:

$$|-3+5| = |2| = 2 < |-3| + |5| = 8$$
$$|3+5| = |8| = |3| + |5|$$
$$|-3-5| = |-8| = 8 = |-3| + |-5|$$

The following statements are all consequences of the definition of absolute value and are often helpful when solving equations or inequalities involving absolute values:

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \le a$ if and only if $-a \le x \le a$
9. $|x| \ge a$ if and only if $x \ge a$ or $x \le -a$

The inequality $|x| < a$ *says* that the distance from x to 0 is less than the positive number a. This means that x must lie between $-a$ and a, as we can see from Figure 1.4.

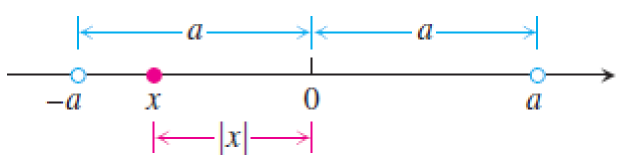


Figure 1.4

Example 4: Solve the equation $|2x - 3| = 7$

Solution:

By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$2x - 3 = 7$	$2x - 3 = -7$	Equivalent equations without absolute values
$2x = 10$	$2x = -4$	Solve as usual.
$x = 5$	$x = -2$	

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$

Example 5: Solve the inequality $\left|5-\frac{2}{x}\right|<1$

Solution We have

$$\left|5-\frac{2}{x}\right|<1 \Leftrightarrow -1<5-\frac{2}{x}<1 \qquad \text{Property 6}$$

$$\Leftrightarrow -6<-\frac{2}{x}<-4 \qquad \text{Subtract 5.}$$

$$\Leftrightarrow 3>\frac{1}{x}>2 \qquad \text{Multiply by } -\frac{1}{2}.$$

$$\Leftrightarrow \frac{1}{3}<x<\frac{1}{2}. \qquad \text{Take reciprocals.}$$

(The symbol <=> is often used by mathematicians to denote the "if and only if" logical relationship. It also means "implies and is implied by.")
The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval $(1/3, 1/2)$.

1.2 Lines, Circles, and Parabolas

1.2.1 Coordinate Geometry and Lines

The points in a plane can be identified with ordered pairs of real numbers. We start by drawing two perpendicular coordinate lines that intersect at the origin O on each line. Usually one line is horizontal with positive direction to the right and is called the ***x*-axis**; the other line is vertical with positive direction upward and is called the ***y*-axis**.
Any point ***P*** in the plane can be located by a unique ordered pair of numbers as follows:
Draw lines through ***P*** perpendicular to the *x*- and *y*-axes. These lines intersect the axes in points with coordinates and as shown in Figure 1.5. Then the point P is assigned the ordered pair (a, b). The first number a is called the ***x*-coordinate** (or **abscissa**) of P; the second number b is called the ***y*-coordinate** (or **ordinate**) of P. We say that P is the point with coordinates (a, b), and we denote the point by the symbol $P\ (a, b)$. Several points are labeled with their coordinates in Figure 1.6.

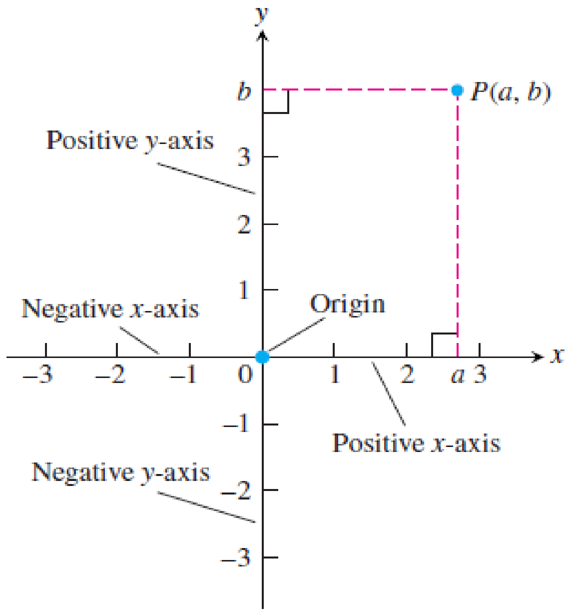


Figure 1.5

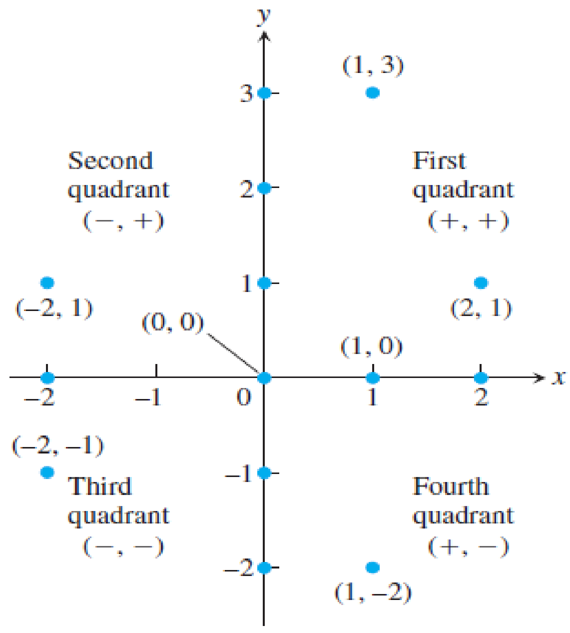


Figure 1.6

This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system**.

The plane supplied with this coordinate system is called the **coordinate plane** or the **Cartesian plane**.

The x- and y-axes are called the coordinate axes and divide the Cartesian plane into four quadrants: First quadrant, Second quadrant, Third quadrant and Fourth quadrant as shown in Figure 1.6. Notice that the First quadrant consists of those points whose x- and y-coordinates are both positive.

Example 6: Describe and sketch the regions given by the following sets:

(a) $\{(x, y) \mid x \geq 0\}$ (b) $\{(x, y) \mid y = 1\}$ (c) $\{(x, y) \mid |y| < 1\}$

Solution:

(a) The points whose x-coordinates are 0 or positive lie on the y-axis or to the right of it as indicated by the shaded region in Figure 1.7 (a).

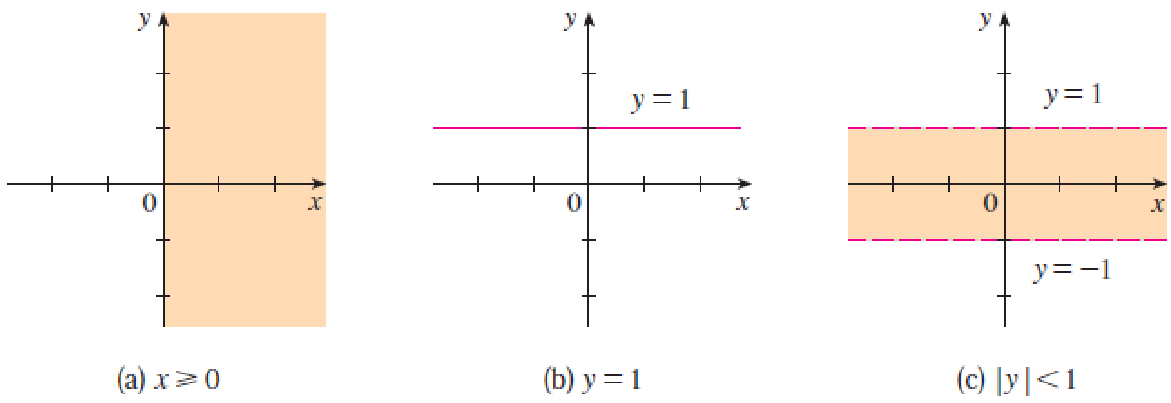


Figure 1.7

(b) The set of all points with y-coordinate 1 is a horizontal line one unit above the x-axis [see Figure 1.7(b)].

(c) $|y| < 1$ if and only if $-1 < y < 1$

The given region consists of those points in the plane whose y-coordinates lie between -1 and 1. Thus the region consists of all points that lie between (but not on) the horizontal lines $y = 1$ and $y = -1$. [These lines are shown as dashed lines in Figure 1.7(c) to indicate that the points on these lines don't lie in the set.]

1.2.2 Increments and Straight Lines

When a particle moves from one point in the plane to another, the net changes in its coordinates are called *increments.* They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. If x changes from x_1 to x_2 the **increment** in x is:

$$\Delta x = x_2 - x_1$$

Example 7: In going from the point $A(4, -3)$ to the point $B(2, 5)$ the increments in the x- and y-coordinates are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8$$

From $C(5, 6)$ to $D(5, 1)$ the coordinate increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5$$

See Figure 1.8

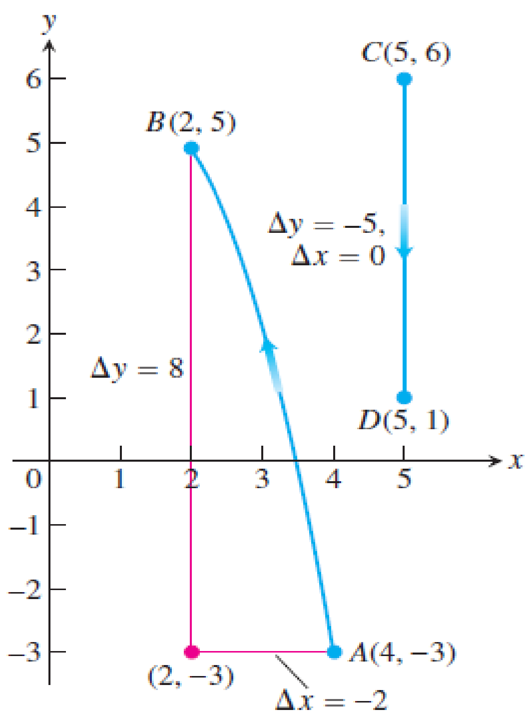


Figure 1.8

1.2.3 Slope of straight line

Slope is a measure of the steepness of the line.
Given two points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the **run** and the **rise**, respectively, between P_1 and P_2. Two such points always determine a unique straight line (usually called simply a line) passing through them both. We call the line P_1 P_2.
Any nonvertical line in the plane has the property that the ratio

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y2-y1}{x2-x1}$$

has the same value for every choice of the two points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$ on the line (Figure 1.9). This is because the ratios of corresponding sides for similar triangles are equal.

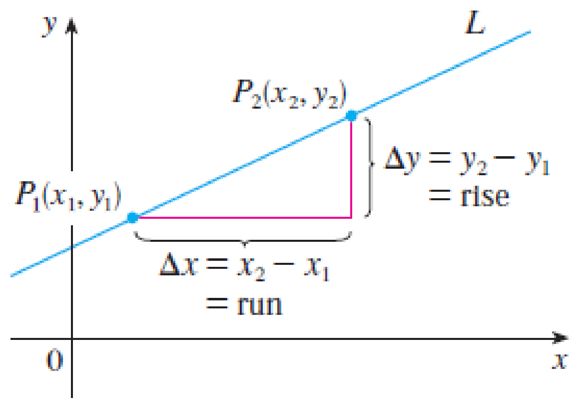


Figure 1.9

DEFINITION The **slope** of a nonvertical line that passes through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a vertical line is not defined.

Figure 1.10 shows several lines labeled with their slopes. Notice that lines with positive slope slant **upward to the right**, whereas lines with negative slope slant **downward to the right**. Notice also that the **horizontal line has slope 0** because $\Delta y = 0$ and the slop of the **vertical line is undefined** because $\Delta x = 0$.

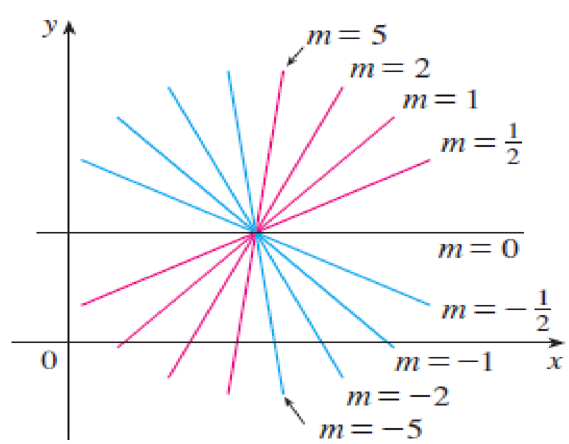


Figure 1.10

Example 8: find the slop of the nonvertical straight line L1 passes through the points $P_1\,(0, 5)$ and $P_2\,(4, 2)$ and L2 passes $P_3\,(0, -2)$ and $P_4\,(3, 6)$.

Solution:

Line *L1*:

The slope of *L1* is $m = \frac{\Delta y}{\Delta x} = \frac{y2-y1}{x2-x1}$

$$= \frac{6-(-2)}{3-0} = \frac{8}{3}$$

That is, *y* increases 8 units every time *x* increases 3 units.

Line *L2*:

The slope of *L2* is $m = \frac{\Delta y}{\Delta x} = \frac{y2-y1}{x2-x1}$

$$m = \frac{\Delta y}{\Delta x} = \frac{2-5}{4-0} = \frac{-3}{4}$$

That is, *y* decreases 3 units every time *x* increases 4 units.

Lines *L1* and *L2* explained in Figure 1.11

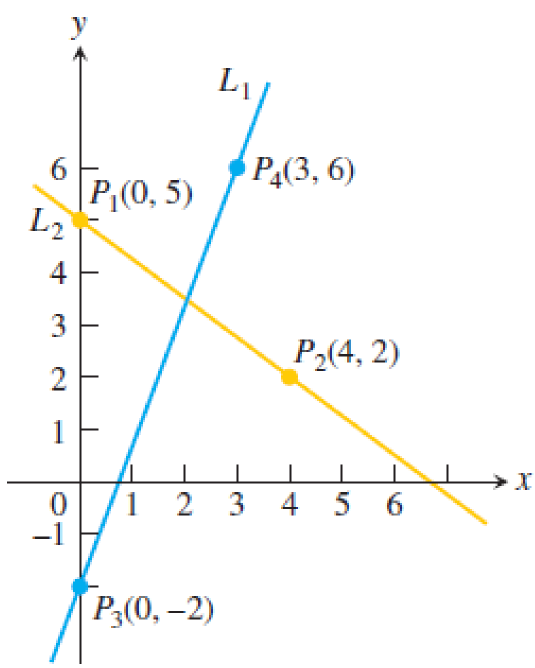


Figure 1.11

1.2.4 Equation of straight line

(a) Point-Slope Form of the Equation of a Line

Now let's find an equation of the line that passes through a given point $P_1\ (x_1, y_1)$ and has slope m. A point $P(x, y)$ with $x \neq x_1$ lies on this line if and only if the slope of the line through P_1 and P is equal to m; that is:

$$\frac{y-y1}{x-x1} = \mathrm{m}$$

This equation can be rewritten in the form:

$$y - y_1 = m\ (x - x_1)$$

and we observe that this equation is also satisfied when $x = x_1$ and $y = y_1$. Therefore it is an equation of the given line.

> The equation
>
> $$y = y_1 + m(x - x_1)$$
>
> is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m.

Example 9: Find an equation of the line through (1, -7) with slope – 1/2.
Solution:
Using Point-slope form of the equation of a line with m = -1/2, x_1 = 1 and y_1= -7, we obtain an equation of the line as:

$$y + 7 = -1/2\ (x - 1)$$

which we can rewrite as:

$$2y + 14 = -\ x + 1 \quad \text{or} \quad x + 2y + 13 = 0$$

Example 10: Write an equation for the line through the point (2, 3) with slope -3/2
Solution:
We substitute x_1 = 2, y_1= 3 and m = -3/2 into the point-slope equation and obtain

$$y = 3 - 3/2\,(x - 2), \quad \text{or} \quad y = -3/2\,(x) + 6$$

When $x = 0$, y = 6 so the line intersects the y-axis at y = 6.

(b) A Line Through Two Points

Example 11: Find an equation of the line through the points (-1, 2) and (3, - 4).
Solution:
By Definition the slope of the line:

$$m = \frac{-4-2}{3-(-1)} = -\frac{3}{2}$$

Using the point-slope form with $x_1 = -1$ and $y_1 = 2$, we obtain:

$$y - 2 = -3/2\,(x + 1)$$

or

$$3x + 2y = 1$$

Example 12: Write an equation for the line through (-2, -1) and (3, 4).
Solution: The line's slope is

$$m = \frac{-1-4}{-2-3} = \frac{-5}{-5} = 1.$$

We can use this slope with either of the two given points in the point-slope equation:

With $(x_1, y_1) = (-2, -1)$	With $(x_1, y_1) = (3, 4)$
$y = -1 + 1 \cdot (x - (-2))$	$y = 4 + 1 \cdot (x - 3)$
$y = -1 + x + 2$	$y = 4 + x - 3$
$y = x + 1$	$y = x + 1$

Same result

Either way, $y = x + 1$ is an equation for the line (Figure 1.12)

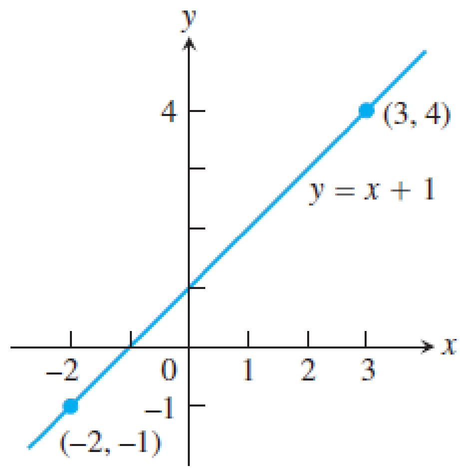


Figure 1.12

(c) Slope-Intercept Form of The Equation of a Line

Suppose a nonvertical line has slope m and y-intercept b. (See Figure 1.13).

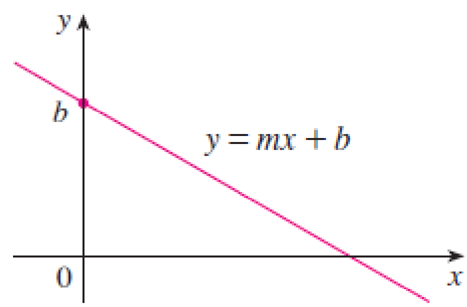


Figure 1.13

This means it intersects the y-axis at the point $(0, b)$, so the point-slope form of the equation of the line, with $x_1 = 0$ and $y_1 = 0$, becomes:

$$y - b = m\,(x - 0)$$

This simplifies as follows:

> The equation
>
> $$y = mx + b$$
>
> is called the **slope-intercept equation** of the line with slope m and y-intercept b.

Example : Find the intercepts of the axis of the equation $y = x^2 - 1$
Solution: For x-intercept, let y = 0 $\rightarrow$ $x^2 - 1 = 0$ $\rightarrow$ $x^2 = 1$ $\rightarrow$ $x = \pm 1$
For y-intercept, let x = 0 $\rightarrow$ $y = 0 - 1$ $\rightarrow$ $y = -1$

In particular, if a line is horizontal, its slope is $m = 0$, so its equation is $y = b$, where b is the y-intercept (see Figure 1.14). A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-coordinate of every point on the line is a.

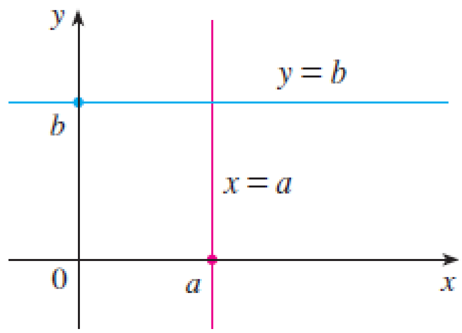


Figure 1.14

Example 13: Write the standard equations for the vertical and horizontal lines through point (2, 3).
Solution:

$(a, b) = (2, 3)$
x-intercept = $a = 2$
y-intercept = $b = 3$

- Horizontal line equation:

$y = m\,x + b$
$\quad = 0\,x + 3$
$y = 3$

- Vertical line equation:

$x = 2$
two lines was shown in Figure 1.15

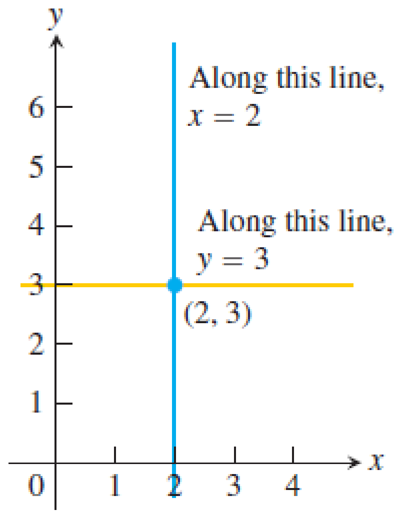


Figure 1.15

(d) The General Equation of a Line

A **linear equation** or the **general equation** of a line can be written in the form:

$$\boxed{Ax + By + C = 0}$$

where, A, B, and C are constants and A and B are not both 0.

We can show that it is the equation of a line:

- If $B = 0$, the equation becomes $Ax + C = 0$ or $x = - C/A$, which represents a vertical line with x-intercept $- C/A$.
- If $B \neq 0$, the equation can be rewritten by solving for y:

$$y = -\frac{A}{B}x - \frac{C}{B}$$

We recognize this as being the **slope-intercept** form of the equation of a line ($m= -A/B$, $b= - C/B$).

Example 14: Find the slope and y-intercept of the line $8x + 5y = 20$
Solution: Solve the equation for y to put it in slope-intercept form:

$$8x + 5y = 20$$
$$5y = -8x + 20$$
$$y = -\frac{8}{5}x + 4.$$

The slope is m = - 8/5. The y-intercept is b = 4

(e) Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular. The following facts are proved:

PARALLEL AND PERPENDICULAR LINES

1. Two nonvertical lines are parallel if and only if they have the same slope.
2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Example 15: Find an equation of the line through the point (5, 2) that is parallel to the line $4x + 6y + 5 = 0$.

Solution: The given line can be written in the form

$$y = -\frac{2}{3}x - \frac{5}{6}$$

which is in slope-intercept form with m = - 2/3. Parallel lines have the same slope, so the required line has slope – 2/3 and its equation in point-slope form is

$$y - 2 = -2/3\,(x - 6)$$

We can write this equation as $2x + 3y = 16$.

Example 16: Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

Solution: The equations can be written as

$$y = -\tfrac{2}{3}x + \tfrac{1}{3} \qquad \text{and} \qquad y = \tfrac{3}{2}x - \tfrac{1}{4}$$

from which we see that the slopes are

$$m_1 = -\tfrac{2}{3} \qquad \text{and} \qquad m_2 = \tfrac{3}{2}$$

Since $m_1 m_2 = -1$, the lines are perpendicular.

1.2.5 Distance and Circles in the Plane

To find the distance $|P_1P_2|$ between any two points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$, we note that triangle $P_1P_2P_3$ in Figure 1.16 is a right triangle, and so by the Pythagorean Theorem we have:

$$|P1P2| = \sqrt{|P1P3|^2 + |P2P3|^2} = \sqrt{|x2 - x1|^2 + |y2 - y1|^2}$$

$$= \sqrt{(x2 - x1)^2 + (y2 - y1)^2}$$

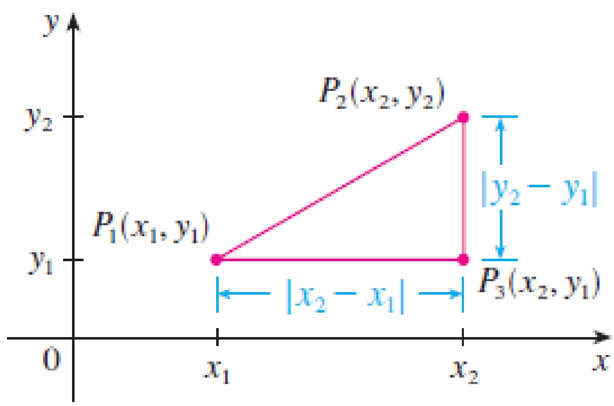


Figure 1.16

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 17: The distance between (1, -2) and (5, 3) is

$$\sqrt{(5-1)^2+[3-(-2)]^2}=\sqrt{4^2+5^2}=\sqrt{41}$$

Second-Degree Equations

In the proceeding sections we saw that a first-degree, or linear, equation $Ax + By + C = 0$ represents a line. In this section we discuss second-degree equations such as

$$x^2 + y^2 = 1 \qquad y = x^2 + 1 \qquad \frac{x^2}{9}+\frac{y^2}{4}=1 \qquad x^2 - y^2 = 1$$

which represent a circle, a parabola, an ellipse, and a hyperbola, respectively.

(a)Circles

To find an equation of the circle with radius r and center (h, k), by definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$is r. (See Figure 1.17).

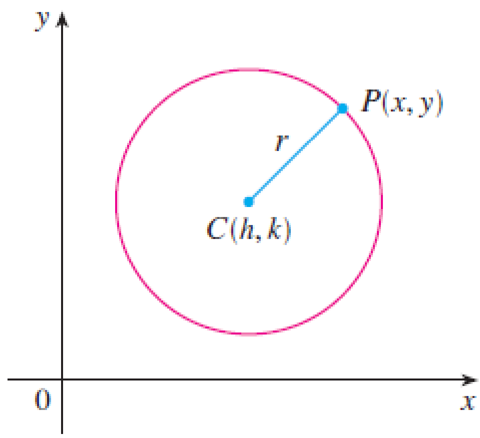


Figure 1.17

Thus P is on the circle if and only if $|PC| = r$. From the distance formula, we have:

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

or equivalently, squaring both sides, we get

$$(x-h)^2 + (y-k)^2 = r^2$$

EQUATION OF A CIRCLE An equation of the circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

In particular, if the center is the origin $(0, 0)$, the equation is

$$x^2 + y^2 = r^2$$

Example 18:

(a) The standard equation for the circle of radius 2 centered at (3, 4) is:

$$(x-3)^2 + (y-4)^2 = 2^2 = 4$$

(b) The circle

$$(x-1)^2 + (y+5)^2 = 3$$

Has $h = 1$, $k = -5$ and $r = \sqrt{3}$. The center is the point $(h, k) = (1, -5)$ and the radius is $r = \sqrt{3}$.

Example 19: Find the center and radius of the circle

$$x^2 + y^2 + 4x - 6y - 3 = 0.$$

Solution: We convert the equation to standard form by completing the squares in x and y:

$$x^2 + y^2 + 4x - 6y - 3 = 0$$ Start with the given equation.

$$(x^2 + 4x \quad) + (y^2 - 6y \quad) = 3$$ Gather terms. Move the constant to the right-hand side.

$$\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) + \left(y^2 - 6y + \left(\frac{-6}{2}\right)^2\right) = 3 + \left(\frac{4}{2}\right)^2 + \left(\frac{-6}{2}\right)^2$$ Add the square of half the coefficient of x to each side of the equation. Do the same for y. The parenthetical expressions on the left-hand side are now perfect squares.

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 3 + 4 + 9$$

$$(x + 2)^2 + (y - 3)^2 = 16$$ Write each quadratic as a squared linear expression.

The center is (-2, 3) and the radius is $r = 4$.

Example 20: Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

SOLUTION We first group the x-terms and y-terms as follows:

$$(x^2 + 2x) + (y^2 - 6y) = -7$$

Then we complete the square within each grouping, adding the appropriate constants to both sides of the equation:

$$(x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9$$

or $$(x + 1)^2 + (y - 3)^2 = 3$$

Comparing this equation with the standard equation of a circle, we see that $h = -1$, $k = 3$ and $r = \sqrt{3}$, so the given equation represents a circle with center (-1, 3) and radius $r = \sqrt{3}$. It is sketched in Figure 1.18.

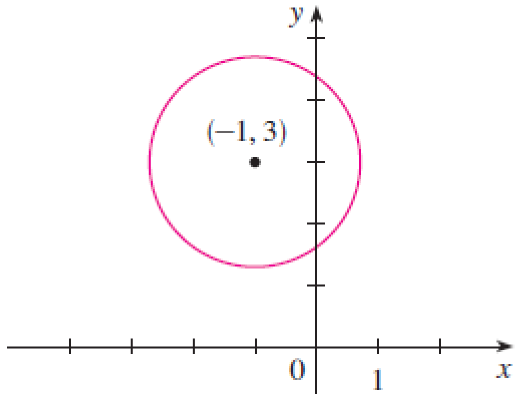


Figure 1.18

(b)Parabola

The geometric properties of parabolas will be reviewed later. Here we regard a parabola as a graph of an equation of the form $y = ax^2 + bx + c$.

Example 21: Draw the graph of the parabola $y = x^2$

Solution:

We set up a table of values, plot points, and join them by a smooth curve to obtain the graph in Figure 1.19.

x	$y = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

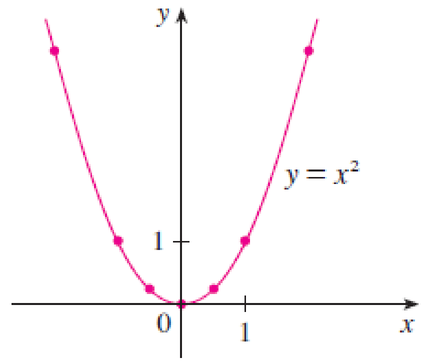


Figure 1.19

Figure 1.20 shows the graphs of several parabolas with equations of the form for various values of the number *a*. In each case the *vertex*, the point where the parabola changes direction, is the origin.

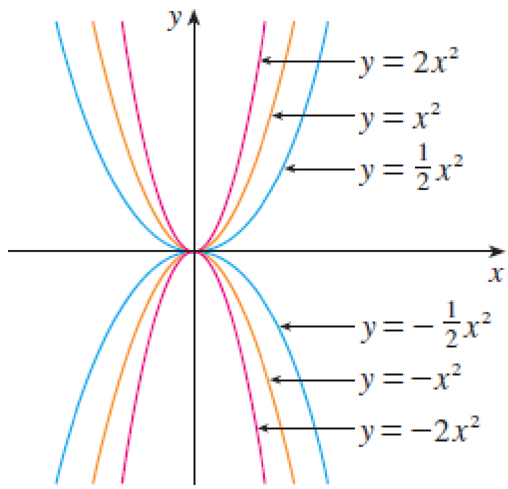


Figure 1.20

We see that:
The parabola $y = ax^2$ opens upward if $a > 0$
The parabola $y = ax^2$ opens downward if $a < 0$. (as in Figure 1.21)

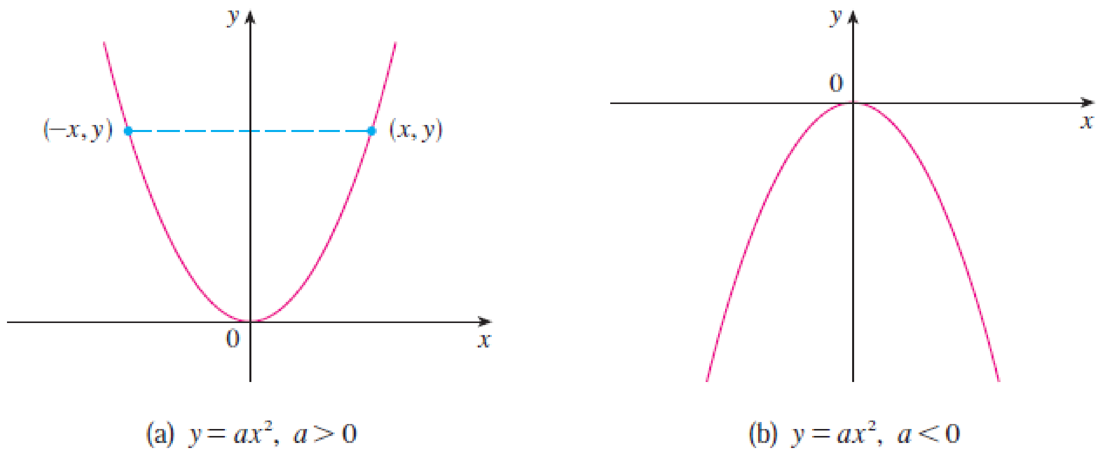


Figure 1.21

Note that:
The graph of an equation is symmetric with respect to the *y*-axis if the equation is unchanged when x is replaced by – *x*, and
The larger the value of | *a* | the narrower the parabola

Generally,

The Graph of $y = ax^2 + bx + c, \quad a \neq 0$
The graph of the equation $y = ax^2 + bx + c, a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The **axis** is the line

$$x = -\frac{b}{2a}. \tag{2}$$

The **vertex** of the parabola is the point where the axis and parabola intersect. Its x-coordinate is $x = -b/2a$; its y-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

Example 22: Graph the equation $-\frac{1}{2}x^2 - x + 4$

Solution: Comparing the equation $y = ax^2 + bx + c$ with we see that

$$a = -\tfrac{1}{2}, \quad b = -1, \quad c = 4$$

Since $a < 0$ the parabola opens downward. From Equation (2) the axis is the vertical line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2\left(-\frac{1}{2}\right)} = -1$$

When $x = -1$, we have

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}$$

The vertex is (-1, 9/2).
The x-intercepts are where $y = 0$:

$$-(1/2)\,x^2 - x + 4 = 0$$
$$x^2 + 2x - 8 = 0$$
$$(x - 2)(x + 4) = 0$$
$$x = 2,\ x = -4$$

We plot some points, sketch the axis, and use the direction of opening to complete the graph in Figure 1.22

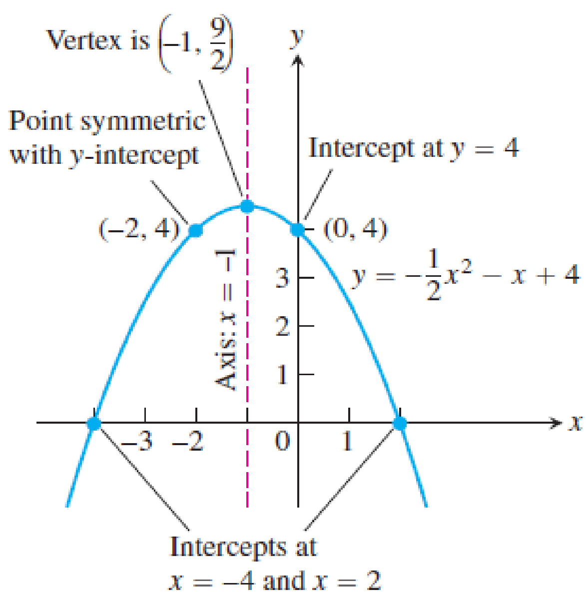
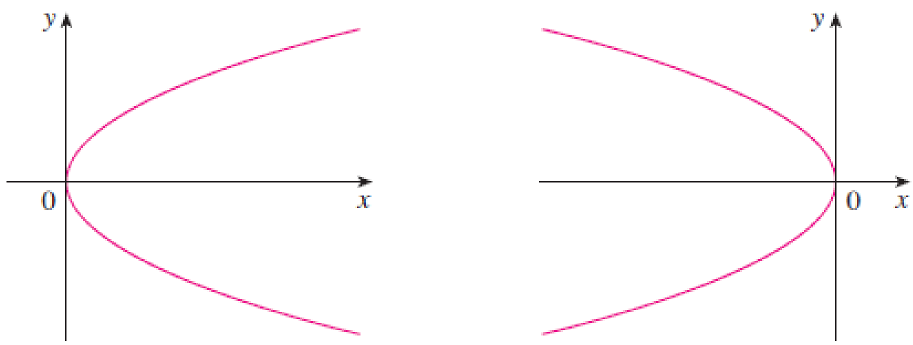


Figure 1.22

If we interchange x and y in the equation $y = ax^2$, the result is $x = ay^2$, which also represents a parabola. The parabola $x = ay^2$ opens to the right if $a > 0$ and to the left if $a < 0$. (See Figure 1.23). This time the parabola is symmetric with respect to the x-axis because if (x, y) satisfies $x = ay^2$, then so does $(x, -y)$.



(a) $x = ay^2,\ a > 0$ (b) $x = ay^2,\ a < 0$

Figure 1.23

> The graph of an equation is symmetric with respect to the x-axis if the equation is unchanged when y is replaced by $-y$.

(c)ELLIPSES

The curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are positive numbers, is called an ellipse in standard position.

The most important properties of the ellipses are:

- The ellipse is symmetric with respect to both axes, i.e the above Equation is unchanged if x is replaced by $-x$ or y is replaced by $-y$.
- The ***x*-intercepts** of a graph are the x-coordinates of the points where the graph intersects the x-axis. They are found by setting $y = 0$ in the equation of the graph.

- The ***y*-intercepts** of a graph are the *y*-coordinates of the points where the graph intersects the *y*-axis. They are found by setting $x = 0$ in the equation of the graph. See Figure 1.24

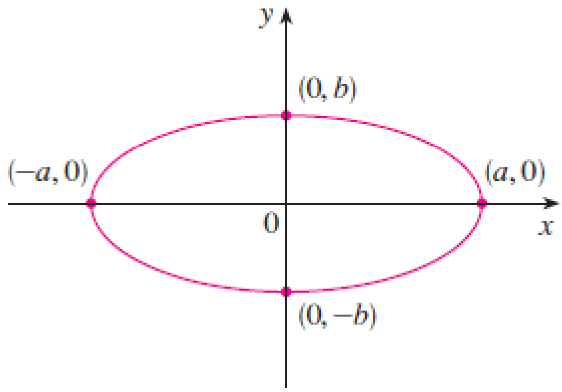


Figure 1.24

Example 23: Sketch the graph of $9x^2 + 16y^2 = 144$.
Solution: We divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$ and $b = 3$. The *x*-intercepts are ± 4; the *y*-intercepts are ± 3. The graph is sketched in Figure 1.25.

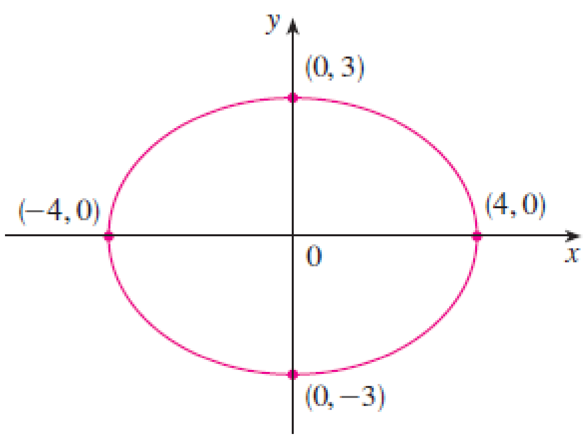


Figure 1.25

(d)HYPERBOLAS

The curve with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where *a* and *b* are positive numbers, is called a hyperbola in standard position.
The most important properties of the hyperbola are:

- The hyperbola is symmetric with respect to both axes, i.e the above Equation is unchanged if x is replaced by $-x$ or y is replaced by $-y$.
- The ***x*-intercepts** of a graph are the *x*-coordinates of the points where the graph intersects the *x*-axis. They are found by setting $y = 0$ in the equation of the graph: $y = 0$ obtain $x^2 = a^2$ and $x = \pm a$.
- If we put $x = 0$ in Equation 3, we get $y^2 = -b^2$, which is impossible, so there is no *y*-intercept.
- The hyperbola consists of two parts, called its branches.
- The hyperbola have two asymptotes, which are the lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 1.26. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. This involves the idea of a limit, which is discussed in proceeding chapters.

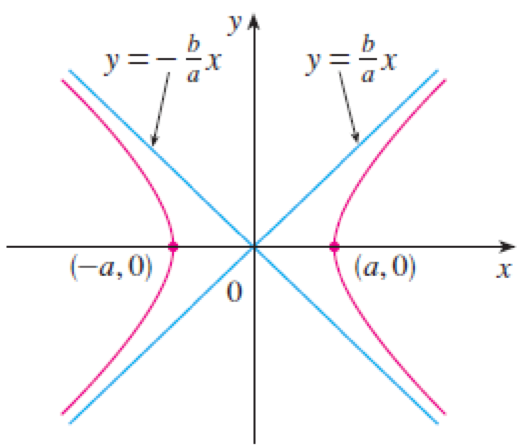


Figure 1.26

By interchanging the roles of and we get an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

which also represents a hyperbola and is sketched in Figure 1.27.

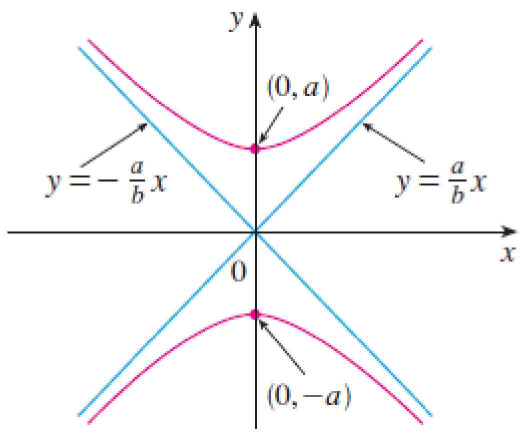


Figure 1.27

Example 24: Sketch the curve $9x^2 - 4y^2 = 36$.
Solution: Dividing both sides by 36, we obtain:

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

which is the standard form of the equation of a hyperbola. Since $a^2 = 4$, the x-intercepts are ± 2. Since $b^2 = 9$, we have b = 3 and the asymptotes are $y = \pm (3/2)\, x$. The hyperbola is sketched in Figure 1.28.

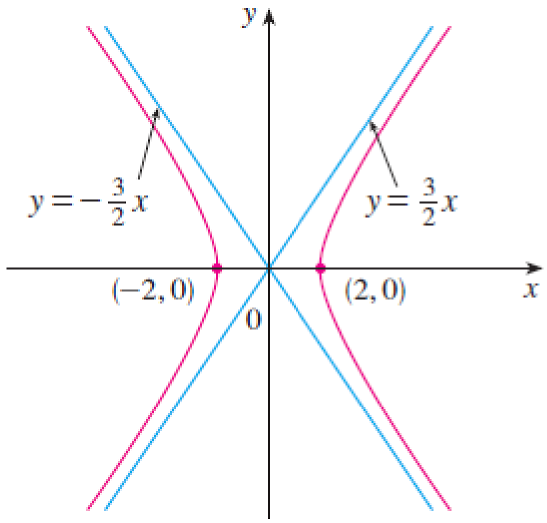
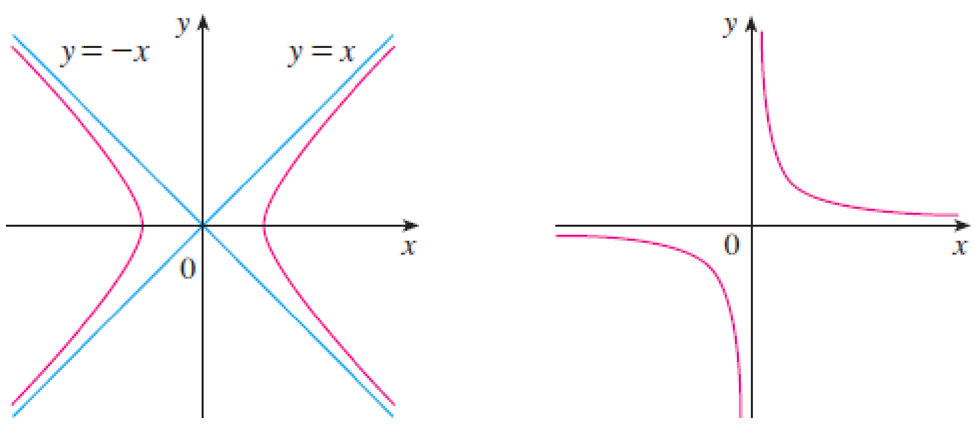


Figure 1.28

If $b = a$, a hyperbola has the equation $x^2 - y^2 = a^2$ (or $y^2 - x^2 = a^2$) and is called an *equilateral hyperbola* [see Figure 1.29(a)]. Its asymptotes are $y = \pm x$, which are perpendicular.

If an equilateral hyperbola is rotated by 45^0, the asymptotes become the x- and y-axes, and it can be shown that the new equation of the hyperbola is $xy = k$, where k is a constant [see Figure 1.29(b)]



(a) $x^2 - y^2 = a^2$ (b) $xy = k$ $(k > 0)$

Figure 1.29

1.3 Functions; Domain and Range

Functions arise whenever one quantity depends on another. A function can be represented by an equation, a graph, a numerical table, or a verbal description.

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets D and E are sets of **real numbers**.
The set D is called the **domain** of the function.
The number *f(x)* is the **value of *f* at *x*** and is read "*f* of *x*."
The **range** of *f* is the set of all possible values of *f(x)* as *x* varies throughout the domain.
A symbol that represents an arbitrary number in the *domain* of a function *f* is called an **independent variable**.
A symbol that represents a number in the *range* of *f* is called a **dependent variable**.
Thus we can think of the **domain** as the set of all possible inputs and the **range** as the set of all possible outputs if we see the function as a kind of machine (Figure 1.30).

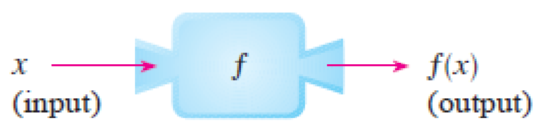


Figure 1.30

Example 25: Verify the domains and associated ranges of the following functions.

(a) $y = x^2$

The formula $y = x^2$ gives a real y-value for any real number x, so the **domain** is $(-\infty, \infty)$. The **range** of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root.

(b) $y = 1/x$

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, **we cannot divide any number by zero**. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$.That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

(c) $y = \sqrt{x}$

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$.
The **domain** of $y = \sqrt{x}$ is $[0, \infty)$
The **range** of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root.

(d) $y = \sqrt{4-x}$
The quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 0$.
The formula gives real y-values for all $x \leq 4$. The **domain** is $(-\infty, 4]$
The **range** of function is $[0, \infty)$, the set of all nonnegative numbers.

(e) $y = \sqrt{1-x^2}$

The **domain** is $[-1, 1]$
The **range** is $[0, 1]$

1.3.1 Graphs of Functions

The most common method for visualizing a function is its graph. If f is a function with domain D, then its graph is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

Example 26: Graph the function $y = x^2$ over the interval [-2, 2].

Solution:

1. Make a table of *xy*-pairs that satisfy the equation $y = x^2$.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

2. Plot the points (x, y) whose coordinates appear in the table (see Figure 1.31)

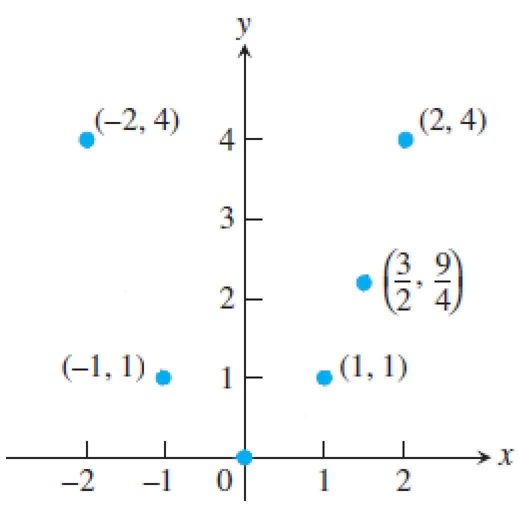


Figure 1.31

3. Draw a *smooth* curve (labeled with its equation) through the plotted points. (Figure 1.32)

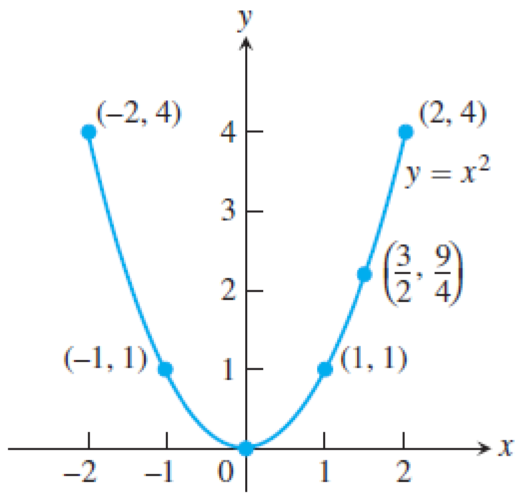


Figure 1.32

1.3.2 Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function.**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Figure 1.33. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$.

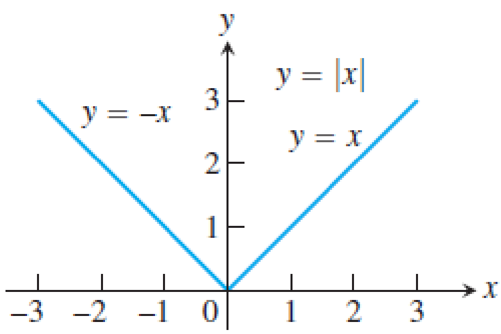


Figure 1.33

Example 27: The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

The values of f are given by:
$y = -x$ when $x < 0$,
$y = x^2$ when $0 \le x \le 1$ and
$y = 1$ when $x > 1$

The function, however, is *just one function* whose domain is the entire set of real numbers (see Figure 1.34)

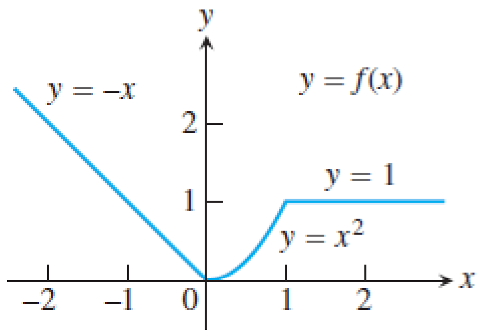


Figure 1.34

Example 28: A function is defined by

$$f(x) = \begin{cases} 1 - x, & x \ge 0 \\ x^2, & x < 0, \end{cases}$$

Evaluate $f(0)$, $f(1)$and $f(2)$ and sketch the graph.
Solution:

Since $0 \le 1$, we have $f(0) = 1 - 0 = 1$
Since $1 \le 1$, we have $f(1) = 1 - 1 = 0$
Since $2 > 1$, we have $f(2) = 2^2 = 4$

See Figure 1.35

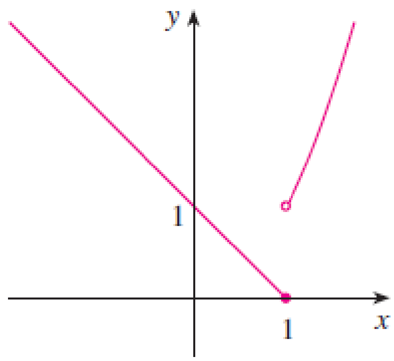


Figure 1.35

1.3.3 Increasing and Decreasing Functions

The graph shown in Figure 1.36 rises from A to B, falls from B to C, and rises again from C to D. The function *f* is said to be increasing on the interval [a, b], decreasing on [b, c], and increasing again on [c, d].

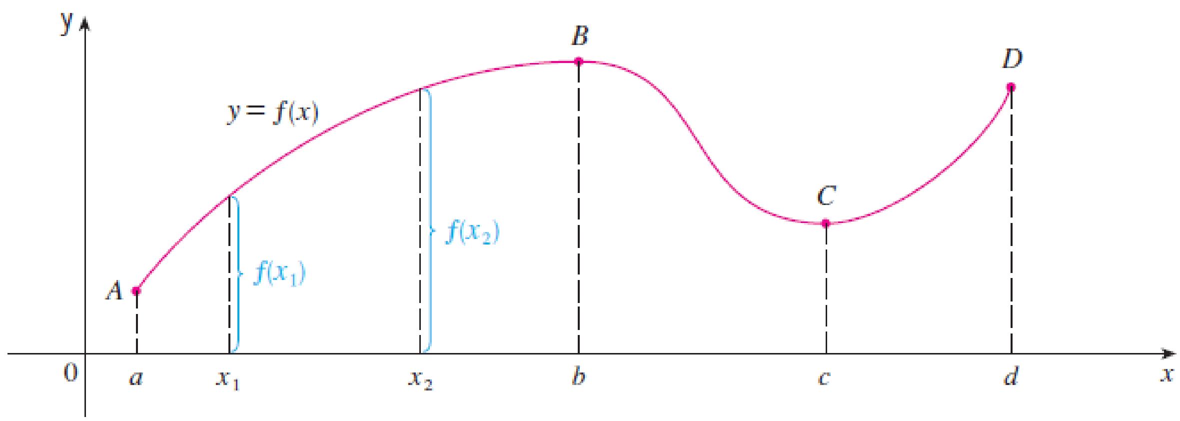


Figure 1.36

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Example 29: investigate the increasing and decreasing intervals of the functions $y = x^2$

Solution:

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_2 < x_1$.

For interval $[0, \infty)$

$f(1) = 1$

$f(2) = 4$

So that, according to the 1st definition the function is increasing on the interval $[0, \infty)$

For interval $(-\infty, 0]$

$f(-1) = 1$

$f(-2) = 4$

The function is decreasing on the interval $(-\infty, 0]$

See Figure 1.37

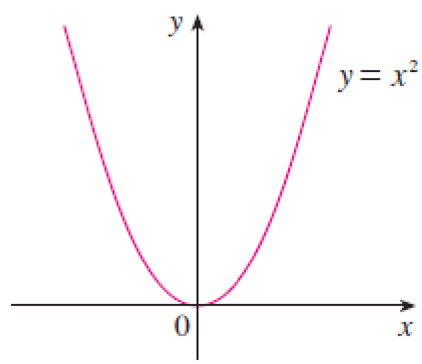


Figure 1.37

1.3.4 Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristics symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

Generally,
The graph of an even function is **symmetric about the *y*-axis**. As for the function $f(x) = x^2$ (see Figure 1.38)
Since always $f(-x) = f(x)$

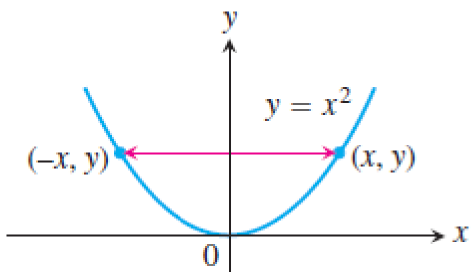


Figure 1.38

The graph of an odd function is **symmetric about the origin**.
For example the function $y = x^3$ (Figure 1.39)
Always $f(-x) = - f(x)$

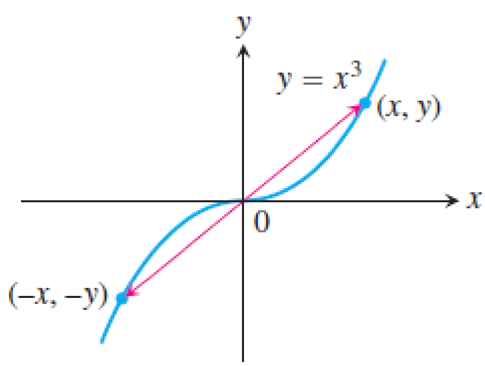


Figure 1.39

Example 30: Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Solution:

(a)

$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore is an odd function.

(b)

$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

so g is even

(c)

$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

The graphs of the functions are shown in Figure 1.40. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

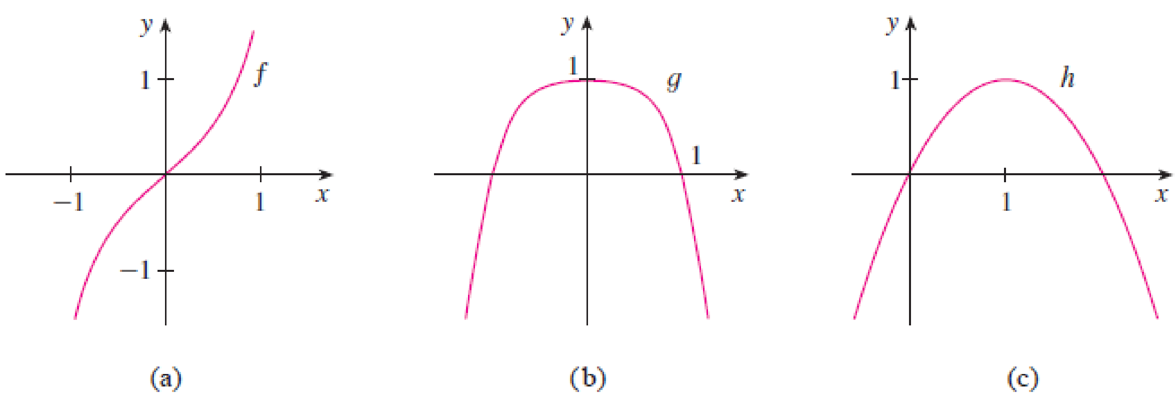


Figure 1.40

1.3.5 Trigonometric Functions

(a) Angles

Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360^o, which is the same as 2π rad. Therefore

$$\boxed{\pi \text{ rad} = 180^\circ}$$

and

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^0 \approx 57.3^o$$

$$1^o = \left(\frac{\pi}{180}\right)^0 \text{rad} \approx 0.017^o$$

Example 31:
(a) Find the radian measure of 60^0.
(b) Express $5\pi/4$ rad in degrees.

Solution:

(a) From above equations we see that to convert from degrees to radians we multiply by $\pi/180$. Therefore

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rad}$$

(b) To convert from radians to degrees we multiply by $180/\pi$. Thus

$$\frac{5\pi}{4} rad = \frac{5\pi}{4}\left(\frac{180}{\pi}\right) = 225°$$

Table 1.2 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.2 Angles measured in degrees and radians

Degrees	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Figure 1.41 shows a sector of a circle with central angle θ and radius r subtending an arc with length a.

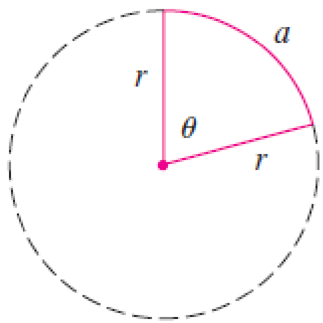


Figure 4.41

Since the length of the arc is proportional to the size of the angle, and since the entire circle has circumference $2\pi r$ and central angle 2π, we have:

$$\frac{\theta}{2\pi} = \frac{a}{2\pi r}$$

Solving this equation for θ and for a, we obtain

$$\boxed{\theta = \frac{a}{r}} \qquad \boxed{a = r\theta}$$

Remember that the above equations are valid only when is measured in **radians**.

Example 32:
(a) If the radius of a circle is 5 cm, what angle is subtended by an arc of 6 cm?
(b) If a circle has radius 3 cm, what is the length of an arc subtended by a central angle of 3π/8 rad?
Solution:

(a) Using Equation $\theta = \frac{a}{r}$ and $a = r\theta$ with a = 6 and r = 5, we see that the angle is

$$\theta = 6/5 = 1.2 \ rad$$

(b) With r = 3 cm and θ = 3π/8 rad, the arc length is:

$$a = r\theta = 3\left(\frac{3\pi}{8}\right) = \frac{9\pi}{8}\text{ cm}$$

(b)standard position

The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive *x*-axis as in Figure 4.42.

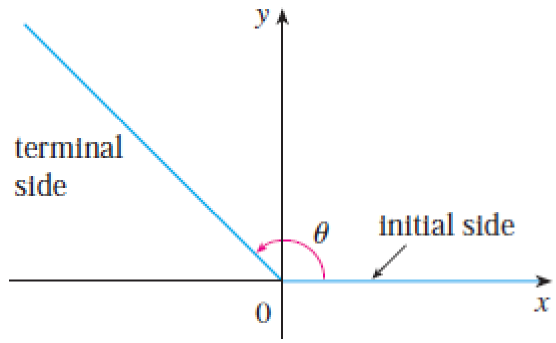


Figure 4.42: $\theta \geq 0$

A **positive** angle is obtained by rotating the initial side **counterclockwise** until it coincides with the terminal side. (as in Figure 4.42)
A negative angle are obtained by **clockwise** rotation as in Figure 4.43

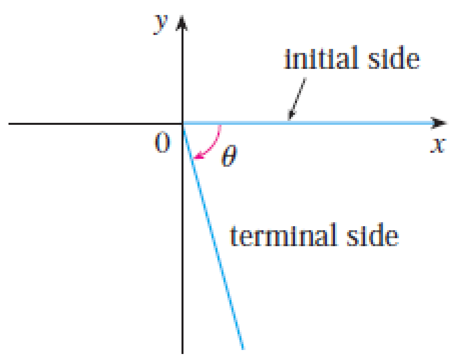


Figure 4.43: $\theta < 0$

Figure 4.44 shows several examples of angles in standard position. Notice that different angles can have the same terminal side. For instance, the angles $3\pi/4$, $-5\pi/4$ and $11\pi/4$ have the same initial and terminal sides because:

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4} \qquad \frac{3\pi}{4} + 2\pi = \frac{11\pi}{4}$$

and 2π rad represents a complete revolution.

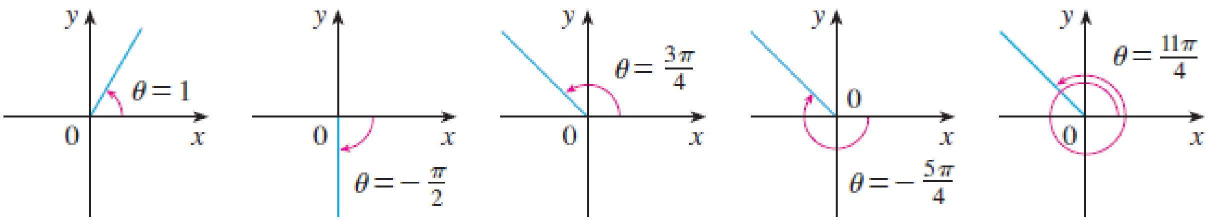


Figure 4.44

(c)The Six Basic Trigonometric Functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows (see Figure 1.45).

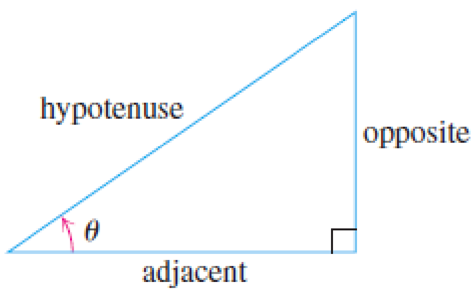


Figure 1.45

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius *r*.

We then define the trigonometric functions in terms of the coordinates of the point *P*(*x*, *y*) where the angle's terminal ray intersects the circle (Figure 1.46).

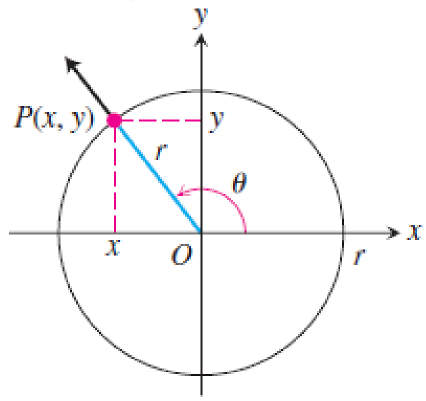


Figure 1.46

$$\textbf{sine:}\quad \sin\theta = \frac{y}{r} \qquad \textbf{cosecant:}\quad \csc\theta = \frac{r}{y}$$

$$\textbf{cosine:}\quad \cos\theta = \frac{x}{r} \qquad \textbf{secant:}\quad \sec\theta = \frac{r}{x}$$

$$\textbf{tangent:}\quad \tan\theta = \frac{y}{x} \qquad \textbf{cotangent:}\quad \cot\theta = \frac{x}{y}$$

These extended definitions agree with the right-triangle definitions when the angle is **acute**.

Notice also that whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

As you can see, ***tan θ*** and ***sec θ*** are not defined if $x = \cos\theta = 0$.

This means they are not defined if θ is $\pm\pi/2, \pm 3\pi/2, \ldots$

Similarly, ***cot θ*** and **csc *θ*** are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

The exact trigonometric ratios for certain angles can be read from the triangles in Figure 1.47. For instance,

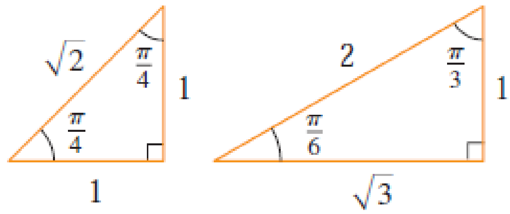


Figure 1.47

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule "All Students Take Calculus" or "CAST" rule shown in Figure 1.48.

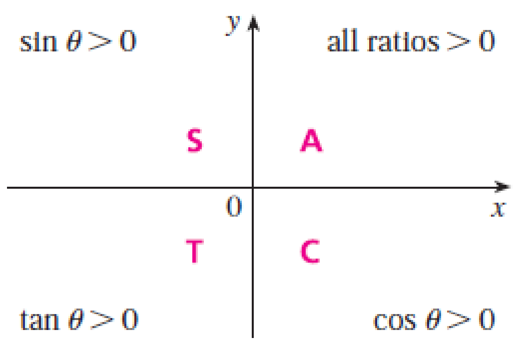


Figure 1.48

Example 33: Find the exact trigonometric ratios for $\theta = 2\pi/3$.
Solution: From the triangle in Figure 1.49 we see that:

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$

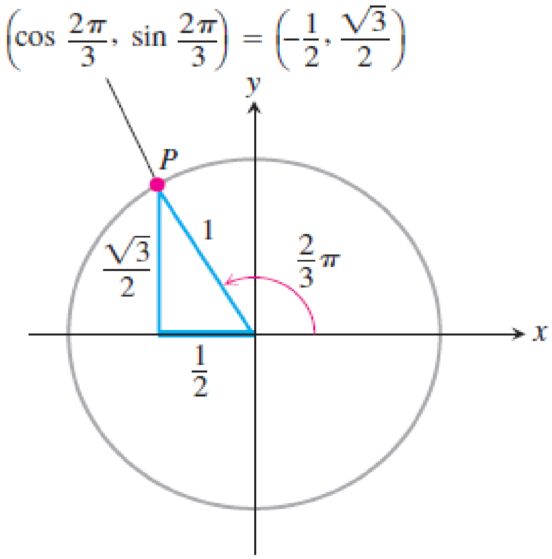


Figure 1.49

Using a similar method we determined the values of sin θ, cos θ, and tan θ shown in Table 1.3

Table 1.3

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
cos θ	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
tan θ	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Example 34: If cos θ = 2/5 and $0 < \theta < \pi/2$, find the other five trigonometric functions of θ.

Solution:

Since cos θ =2/5, we can label the hypotenuse as having length 5 and the adjacent side as having length 2 in Figure 1.50. If the opposite side has length x, then the Pythagorean Theorem gives $x^2 + 4 = 25$ and so $x^2 = 21$, $x = \sqrt{21}$.

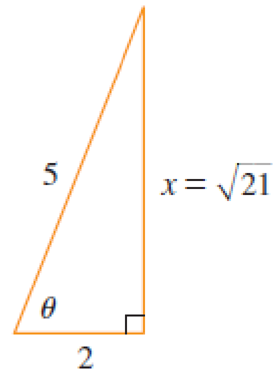


Figure 1.50

We can now use the diagram to write the other five trigonometric functions:

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2} \qquad \cot\theta = \frac{2}{\sqrt{21}}$$

Example 35: Use a calculator to approximate the value of x in Figure 1.51.

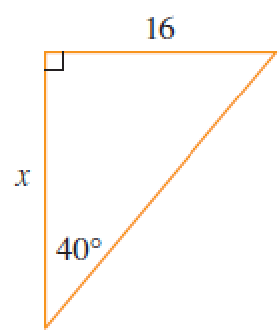


Figure 1.51

Solution: From the diagram we see that:

$$\tan 40° = \frac{16}{x}$$

Therefore,

$$x = \frac{16}{\tan 40°} \approx 19.07$$

(d)Trigonometric Identities

A trigonometric identity is a relationship among the trigonometric functions. The most elementary are the following:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad (1)$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad (2)$$

For the next identity we refer back to Figure 1.46. The distance formula (or, equivalently, the Pythagorean Theorem) tells us that $x^2 + y^2 = r^2$. Therefore

$$sin^2\theta + cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

We have therefore proved one of the most useful of all trigonometric identities:

$$\boxed{\cos^2\theta + \sin^2\theta = 1. \qquad (3)}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry.
Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives:

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The identity

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

Show that **sin** is an odd function and **cos** is an even function.

Since the angles θ and $\theta + 2\pi$ have the same terminal side, we have:

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

The remaining trigonometric identities are all consequences of two basic identities called the **addition formulas**:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

By substituting – *y* for in above equations and using equations [$\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$] we obtain the following **subtraction formulas**:

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Then, by dividing the formulas in **addition formulas** or **subtraction formulas**, we obtain the corresponding formulas for $\tan(x \pm y)$:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

If we put $y = x$ in the addition formulas, we get the **double-angle formulas**:

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

Then, by using the identity $\sin^2 x + \cos^2 x = 1$, we obtain the following alternate forms of the double-angle formulas for $\cos 2x$:

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = 1 - 2\sin^2 x$$

If we now solve these equations for $\cos^2 x$ and $\sin^2 x$, we get the following **half-angle formulas**, which are useful in integral calculus:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Finally, we state the **product formulas**, which can be deduced from **addition and subtraction formulas**:

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

Example 36 Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$.
Solution: Using the double-angle formula, we rewrite the given equation as

$$\sin x = 2 \sin x \cos x \qquad \text{or} \qquad \sin x(1 - 2\cos x) = 0$$

Therefore, there are two possibilities:

$$\sin x = 0 \qquad \text{or} \qquad 1 - 2\cos x = 0$$

$$x = 0, \pi, 2\pi \qquad\qquad \cos x = \tfrac{1}{2}$$

$$x = \pi/3,\ 5\pi/3$$

The given equation has five solutions:0, π/3, π, 5π/3, and 2π.

(e)Periodicity and Graphs of the Trigonometric Functions

The graph of the trigonometric function is obtained by plotting points for one period and then using the periodic nature of the function to complete the graph.

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic:*

Periods of Trigonometric Functions

Period π: $\tan(x+\pi) = \tan x$

$\cot(x+\pi) = \cot x$

Period 2π: $\sin(x+2\pi) = \sin x$

$\cos(x+2\pi) = \cos x$

$\sec(x+2\pi) = \sec x$

$\csc(x+2\pi) = \csc x$

Example 37: plot the trigonometric function ***sin x***.

Solution:
The function will be: $y = f(x) = \sin(x)$
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$

Make a table for xy values

x	***y = sin (x)***
0	0
π/2	1
π	0
3 π/2	-1
2 π	0
- π/2	-1
- π	0

The plot of *sin x* was shown in Figure 1.52. The function *y = sin x* is an odd function (*f (-x) = - f (x)*)

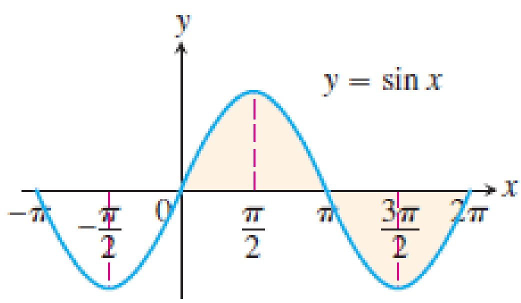
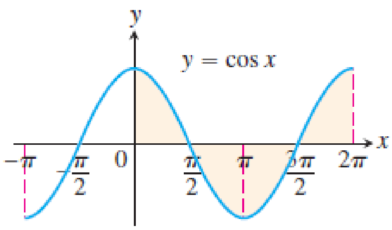


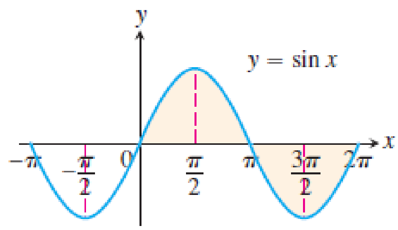
Figure 1.52

Figure 1.53 shows the graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.



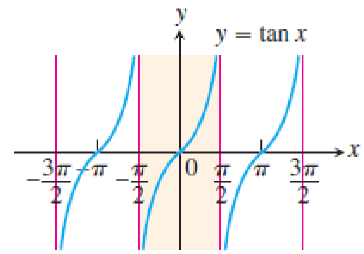
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(a)



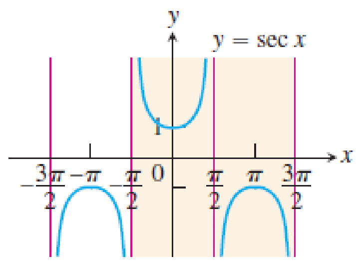
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(b)



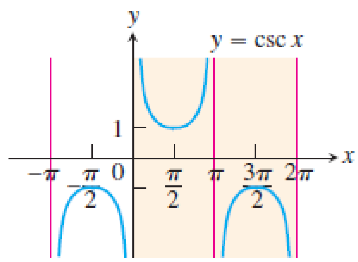
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$
Range: $-\infty < y < \infty$
Period: π

(c)



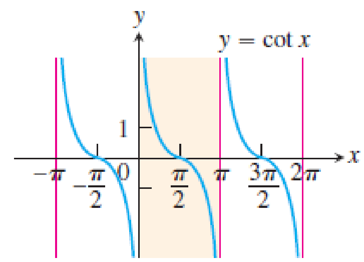
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

(d)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \dots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \dots$
Range: $-\infty < y < \infty$
Period: π

(f)