CHAPTER 4
Applications of Derivatives

4.1 Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem.*

Example 1: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?
Solution: Figure 1 shows a partially filled conical tank. The variables in the problem are:

V = volume (ft^3) of the water in the tank at time t (min)
x = radius (ft) of the surface of the water at time t
y = depth (ft) of the water in the tank at time t.

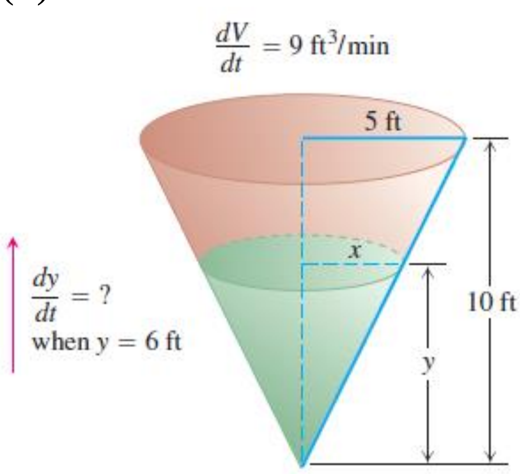


Figure 1

We assume that V, x, and y are differentiable functions of t. The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 6 \text{ ft} \quad \text{and } dV/dt = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. The similar triangles in Figure 1 give us a way to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10} \text{ or } x = \frac{y}{2}$$

Therefore, find

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

To give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12}.3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for dy/dt.

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft /min.

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).

4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

Example 2: A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is π/4, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps:

1. *Draw a picture and name the variables and constants* (Figure 2). The variables in the picture are:

θ = the angle in radians the range finder makes with the ground.

y = the height in feet of the balloon.

We let t represent time in minutes and assume that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

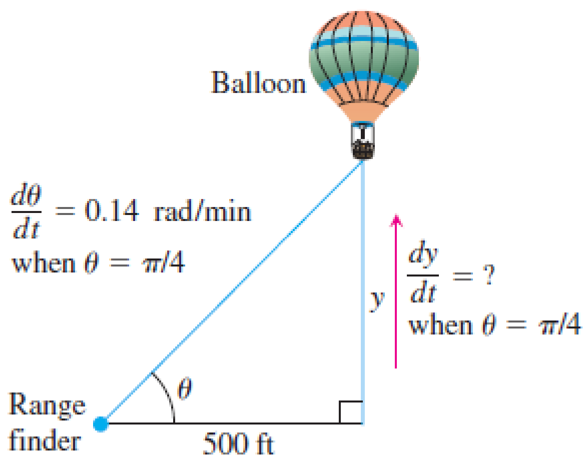


Figure 2

2. *Write down the additional numerical information.*

$$d\theta/dt = 0.14 \text{ rad/min} \qquad \text{when } \theta = \pi/4$$

3. *Write down what we are to find.* We want dy/dt when $\theta = \pi/4$
4. *Write an equation that relates the variables y and θ*

$$y/500 = \tan\theta \quad \text{or } y = 500 \tan\theta$$

5. *Differentiate with respect to t using the Chain Rule.* The result tells how (which we want) is related to (which we know).

$$dy/dt = 500\,(\sec^2\theta)\, d\theta/dt$$

6. *Evaluate with θ = π/4 and dθ/dt = 0.14 to find dy/dt.*

$$dy/dt = 500\,(\sqrt{2})^2(0.14) = 140 \quad [\sec \pi/4 = \sqrt{2}]$$

At the moment in question, the balloon is rising at the rate of 140 ft/min.

Example 3: A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution: We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Figure 3).
We let t represent time and set

x = position of car at time t
y = position of cruiser at time t
s = distance between car and cruiser at time t .

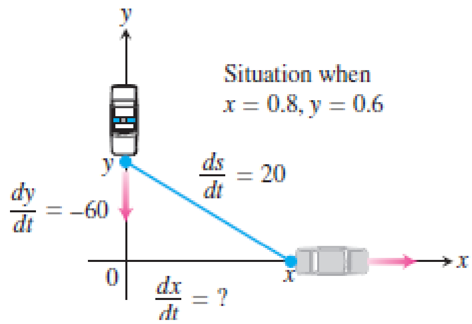


Figure 3

We assume that x, y, and s are differentiable functions of t.
We want to find dx/dt when

$x = 0.8$ mi, $y = 0.6$ mi, $dy/dt = -60$ mph, $ds/dt = 20$ mph.

Note that dy/dt is negative because y is decreasing.
We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph.

Example 4: A water trough is 10 m long and a cross section has the shape of isosceles trapezoid as shown in Figure 4. If the trough is being filled with water at

the rate of 0.2 m^3/ min, how fast is the water level rising when the water is 30 cm deep?

Solution:

V = volume of water

$V = \frac{(0.3+2x)+0.3}{2} * h * 10 = (0.6 + 2x) * 5h = 3h + 10xh$

From similarity of triangles (Figure 4):

$\frac{x}{h} = \frac{0.25}{0.5} = \frac{1}{2}$

$x = h/2$

$V = 3\frac{dh}{dt} + 10\,h\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{dv/dt}{3+10h}$

At $h = 30$ cm $= 0.3$ m $\quad \frac{dh}{dt} = \frac{0.2}{3+10(0.3)} = \frac{0.2}{6} = \frac{1}{30}$ m/min

Example 5: A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of in order to keep the aircraft within its direct line of sight?

Solution: The aircraft *A* and radar station *R* are pictured in the coordinate plane, using the positive *x*-axis as the horizontal distance at sea level from *R* to *A*, and the positive *y*-axis as the vertical altitude above sea level. We let *t* represent time and observe that $y = 12000$ is a constant. The general situation and line-of-sight angle θ are depicted in Figure 5. We want to find dx/dt when $\theta = \pi/6$ rad and $d\theta/dt = 2/3$ deg/sec.

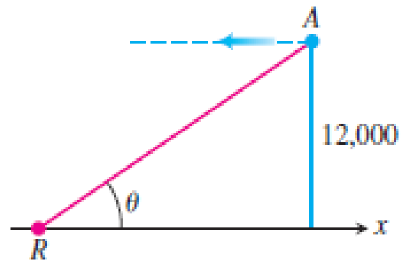


Figure 5

From Figure 5, we see that

$$12{,}000/\,x = \tan\theta \quad \text{or } x = 12{,}000 \cot\theta$$

Using miles instead of feet for our distance units, the last equation translates to

$$x = \frac{12{,}000}{5280}\cot\theta.$$

Differentiation with respect to *t* gives

$$\frac{dx}{dt} = -\frac{12{,}00}{528}\csc^2\theta\,\frac{d\theta}{dt}$$

When $\theta = \pi/6$, $\sin^2\theta = 1/4$,so $\csc^2\theta = 4$. Converting $d\theta/dt = 2/3\ deg/sec$ to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{2}{3}\left(\frac{\pi}{180}\right)(3600)\ \text{rad/hr} \qquad [\ 1\ \text{hr} = 3600\ \text{sec},\ 1\ \text{deg} = \pi/180\ \text{rad}]$$

Substitution into the equation for dx/dt then gives

$$\frac{dx}{dt} = \left(-\frac{1200}{528}\right)(4)\left(\frac{2}{3}\right)\left(\frac{\pi}{180}\right)(3600) \approx -380$$

The negative sign appears because the distance *x* is decreasing, so the aircraft is approaching the island at a speed of approximately 380 mi/hr when first detected by the radar.

Example 6: Figure 6 (a) shows a rope running through a pulley at *P* and bearing a weight *W* at one end. The other end is held 5 ft above the ground in the hand *M* of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the

worker is walking rapidly away from the vertical line *PW* at the rate of 6 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from *PW*?

Solution: We let *OM* be the horizontal line of length x ft from a point *O* directly below the pulley to the worker's hand *M* at any instant of time (Figure 6). Let h be the height of the weight *W* above *O*, and let z denote the length of rope from the pulley *P* to the worker's hand. We want to know dh/dt when $x = 21$ given that $dx/dt = 6$.

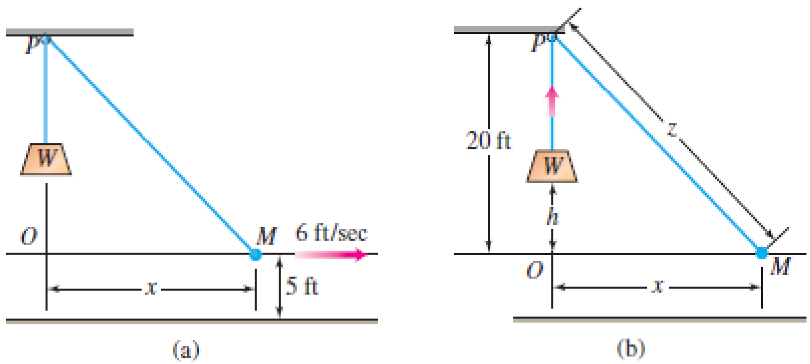


Figure 6

Note that the height of *P* above *O* is 20 ft because *O* is 5 ft above the ground. We assume the angle at *O* is a right angle.

At any instant of time t we have the following relationships (see Figure 5b):

$$20 - h + z = 45 \quad \text{[Total length of rope is 45 ft]}$$

$$20^2 + x^2 = z^2 \quad \text{[Angle at } O \text{ is a right angle]}$$

If we solve for $z = 25 + h$ in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2 \quad \ldots\ldots (1)$$

Differentiating both sides with respect to t gives

$$2x\frac{dx}{dt} = 2(25 + h)\frac{dh}{dt}$$

and solving this last equation for dh/dt we find

$$\frac{dh}{dt} = \left(\frac{x}{25+h}\right)\frac{dx}{dt} \quad \ldots\ldots\ (2)$$

Since we know dx/dt, it remains only to find $25 + h$ at the instant when $x = 21$. From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

So that

$$(25 + h)^2 = 841 \quad \text{or} \quad 25 + h = 29$$

Equation (2) now gives

$$dh/dt = (21/29)*6 \approx 4.3 \text{ ft/sec}$$

as the rate at which the weight is being raised when $x = 21$ ft

4.2 Extreme Values of Functions

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Once we can do this, we can solve a variety of problems in which we find the optimal (best) way to do something in a given situation. Finding maximum and minimum values is one of the most important applications of the derivative.

Definitions: Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Note: Functions with the same defining rule or formula can have different extrema (maximum or minimum values), depending on the domain.
For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 7).

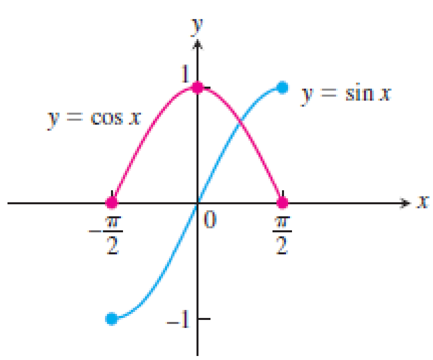


Figure 7

The Extreme Value Theorem: If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$ and $m \le f(x) \le M$ for every other x in $[a, b]$.

4.2.1 The Mean Value Theorem

Rolle's Theorem Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f(c) = 0$. (Figure 8)

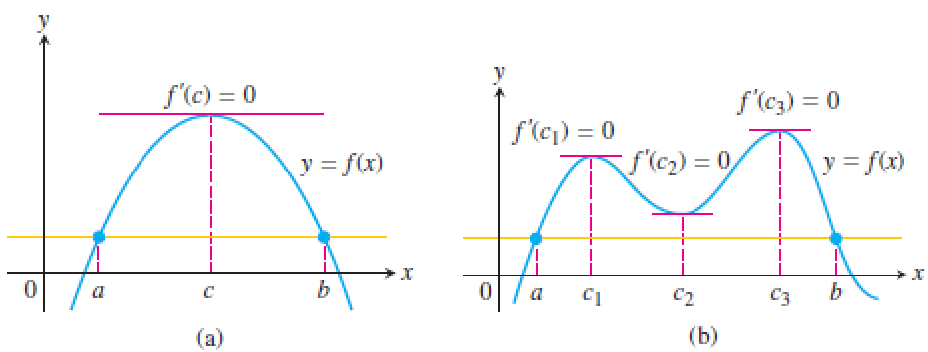


Figure 8

4.2.2 The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b) (Figure 9). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f(c)$$

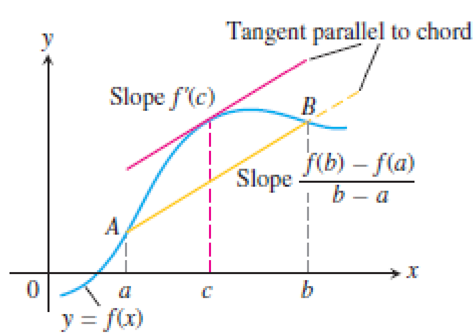


Figure 9

Example 7: if $f(x) = x^2$, $0 \leq x \leq 2$. Find c by using the mean value theorem.

Solution: $f(x) = 2x$, $f(0) = 0$, $f(2) = 4$

$$f(c) = \frac{f(b) - f(a)}{b - a} = \frac{4 - 0}{2 - 0} = 2$$

$f(c) = 2c$

$2c = 2$

$c = 1$

See Figure 10

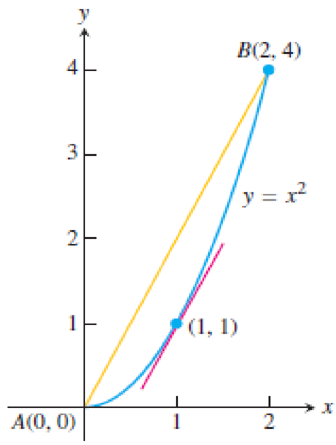


Figure 10

Example 8: To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$.

Solution:

Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on [0, 2] and differentiable on (0, 2).

Therefore, by the Mean Value Theorem, there is a number c in (0, 2) such that

$$f(2) - f(0) = f(c)(2 - 0)$$

Now $f(2) = 6$, $f(0) = 0$, and $f(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)\,2 = 6c^2 - 2$$

which gives $c^2 = 4/3$, that is, $c = \mp 2/\sqrt{3}$. But c must lie in (0, 2), so $c = 2/\sqrt{3}$.

Figure 11 illustrates this calculation: The tangent line at this value of c is parallel to the secant line OB.

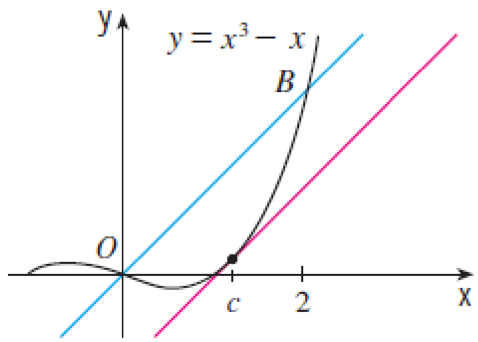


Figure 11

4.2.3 Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval.

4.2.4 Increasing Functions and Decreasing Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
If $f(x) > 0$ at each point $x \in (a, b)$, then f is increasing on [a, b].
If $f(x) < 0$ at each point $x \in (a, b)$, then f is increasing on [a, b].
If $f(x) = 0$ at each point $x \in (a, b)$, then f is critical point (may be max. or min).

Example 9: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing.

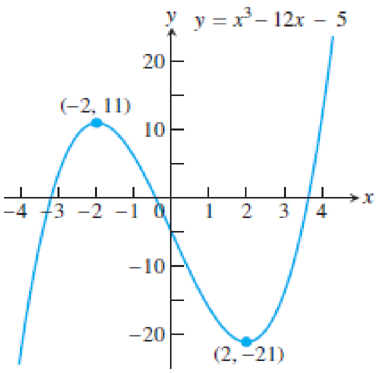
Solution: The function f is everywhere continuous and differentiable. The first derivative

$$f(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x + 2)(x - 2)$$

is zero at $x = -2$ and $x = 2$.
These critical points subdivide the domain of f to create nonoverlapping open intervals:
$(-\infty, -2)$
$(-2, 2)$
$(2, \infty)$
on which f is either positive or negative.

We determine the sign of f by evaluating f at a convenient point in each subinterval. The behavior of f is determined by then applying the above definition to each subinterval.

The results are summarized in the following table, and the graph of f is given in Figure 12.



Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

Figure 12

4.2.5 First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself.

Moving across this interval from left to right,

1. if f changes from negative to positive at c, then f has a local minimum at c;

2. if f changes from positive to negative at c, then f has a local maximum at c;

3. if f does not change sign at c (that is, is positive on both sides of c or negative on both sides), then f has no local extremum at c.

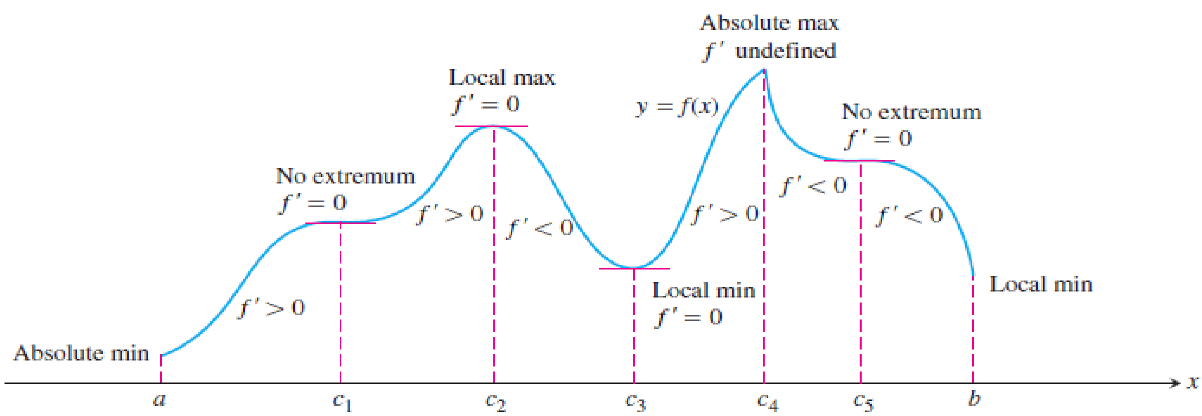


Figure 13

Example 10: Find the critical points of

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$$

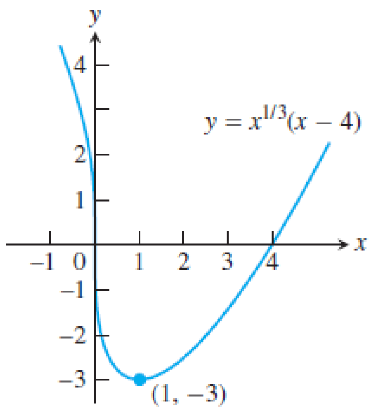
Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution: The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x-4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}\left(x-1\right) = \frac{4(x-1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$ (see Figure 14).



Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Figure 14

4.2.6 Concavity and Curve Sketching

4.2.6a Concavity

Definition: The graph of a differentiable function $y = f(x)$ is
(a) Concave up on an open interval I if f is increasing on I;
(b) Concave down on an open interval I if f is decreasing on I.

4.2.6b The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f > 0$ on I, the graph of f over I is **concave up**.
2. If $f < 0$ on I, the graph of f over I is **concave down**.

Example11

(a) The curve (Figure 15) is concave down on $(-\infty, \infty)$ where $y = 6x < 0$ and concave up on $(0, \infty)$ where $y = 6x > 0$.

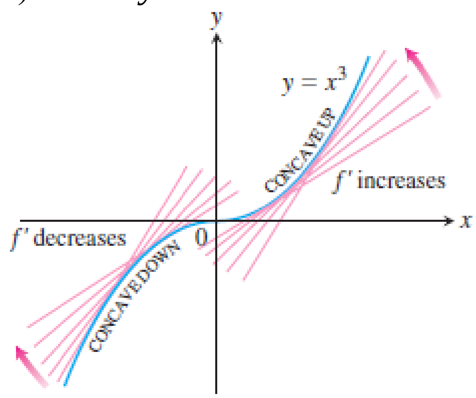


Figure 15

(b) The curve $y = x^2$ (Figure 16) is concave up on $(-\infty, \infty)$ because its second derivative $y = 2$ is always positive.

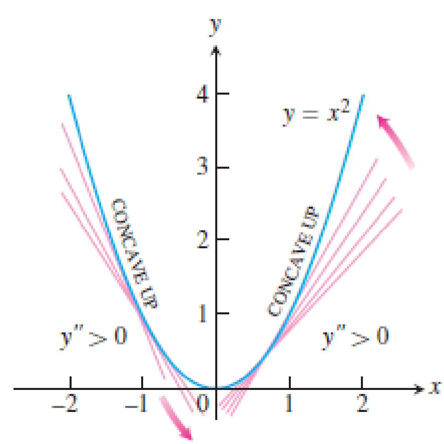


Figure 16

Example 12: Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y = \cos x$ and the second derivative is $y = - \sin x$

The graph of $y = 3 + \sin x$ is concave down on (0, π), where $y = -\sin x$ is negative. It is concave up on (π, 2π), where $y = -\sin x$ is positive (Figure 17).

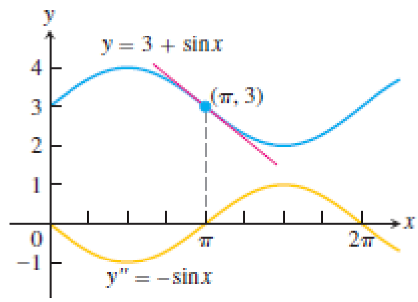


Figure 17

4.2.6c Points of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Note: At a point of inflection (c, f(c)), either f (c) = 0 or f(c) fails to exist.

Example 13: The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f(x) = (5/3)\, x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f(x) = \frac{d}{dx}\left(\frac{5}{3}x^{\frac{2}{3}}\right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 18.

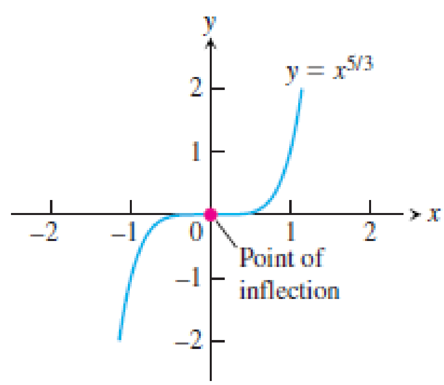


Figure 18

Example 14: The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 19). Even though the second derivative $y = 12\, x^2$ is zero there, it does not change sign.

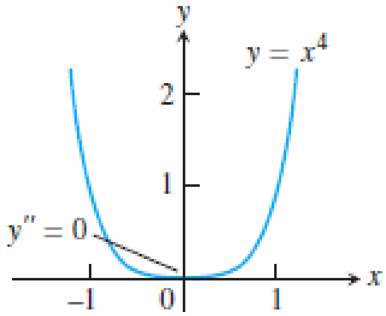


Figure 19

Example 15: The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y = \frac{d^2}{dx^2}\left(x^{\frac{1}{3}}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-\frac{2}{3}}\right) = -\frac{2}{9}x^{-5/3}$$

However, both $y = x^{-2/3}\,/3$ and y fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 20.

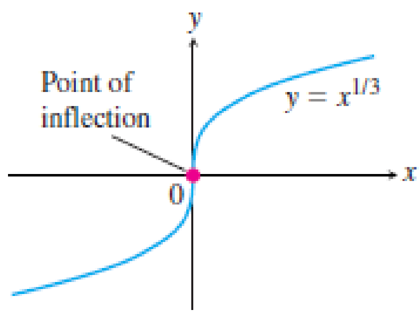


Figure 20

4.2.7 Second Derivative Test for Local Extrema

Suppose f is continuous on an open interval that contains $x = c$.
1. If $f\,(c) = 0$ and $f\,(c) < 0$, then f has a local maximum at $x = c$.
2. If $f\,(c) = 0$ and $f\,(c) > 0$, then f has a local maximum at $x = c$.
3. If $f\,(c) = 0$ and $f\,(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

4.2.8 Curve Sketching

Procedure for Graphing *y = f (x)*

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y and y.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Example 16: Sketch the curve $y = (x - 2)^3 + 1$
Solution:

1. Domain = $(-\infty, \infty)$

Symmetry: $f(x) = y = x^3 - 6x^2 + 12x - 8$
$f(-x) = y = (-x)^3 - 6(-x)^2 + 12(-x) - 8$
$f(x) = y = -x^3 - 6x^2 - 12x - 8$
$f(x) = y = -(x^3 + 6x^2 - 12x + 8)$

$f(-x) \neq f(x)$ and $f(x) \neq -f(x)$
the function nor odd or even

2. $y = 3(x - 2)^2 (1) + 0 = 3(x - 2)^2$

$y = 0 \quad 3(x - 2)^2 = 0 \quad 3(x - 2)(x - 2) = 0 \quad x = 2$
no maximum or minimum point at $x = 2$

$y = 6(x - 2) = 6x - 12$
$y = 0 \quad 6x - 12 = 0 \quad x = 2$
at $x = 2$ inflection point
at $x = 2 \quad y = (2 - 2)^3 + 1 = 1$
(2, 1) is inflection point

3. Intercepts

- For x – intercept let $y = 0$

$0 = (x - 2)^3 + 1$
By inspection, $y = 0$ *if* $x = 1$
(1, 0) is the x – intercept

- For y – intercept, let $x = 0$

$y = (0 - 2)^3 + 1 = -8 + 1 = -7$
(0, -7) is the y - intercept

Example 17: sketch the curve $y = \frac{1}{4} x^4 - x^3 + 4x + 2$
Solution:

1. $y = x^3 - 3x^2 + 4 + 0$

$y = 0 \qquad x^3 - 3x^2 + 4 = 0$
if $x = -1 \qquad (-1)^3 - 3(-1)^2 + 4 = 0$
$(x + 1)(x^2 - 4x + 4) = 0$
$(x + 1) = 0 \qquad x = -1$
$x^2 - 4x + 4 = 0 \qquad (x - 2)(x - 2) = 0 \qquad x = 2$

At $x = -1$ local min. point

$y = \frac{1}{4} + 1 - 4 + 2$
$= -\frac{3}{4}$
(-1, -3/4) is min. point

2. $y = 3x^2 - 6x$

$y = 0 \qquad 3x^2 - 6x = 0 \qquad 3x(x - 2) = 0$
$3x = 0 \quad x = 0$
$x - 2 = 0 \quad x = 2$
(0, 2) (2, 6) are inflection points

3. Intercepts:
   - y-intercept: $x = 0$

$y = 0 - 0 + 0 + 2 = 2$
(0, 2) is y-intercept

Example 18: Sketch $y = \sin x + \cos x$ from $x = -\pi/4$ to $3\pi/4$

Solution:

$y = \cos x - \sin x$

$y = 0 \quad \sin x = \cos x \quad \tan x = 1 \quad x = \pi/4$

at $x = \pi/4 \quad y = \sin \pi/4 + \cos \pi/4 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$(\pi/4, \sqrt{2})$ is max. point

2. $y = -\sin x - \cos x$

$-\sin x - \cos x = 0 \quad \tan x = -1 \quad x = -\pi/4$ and $x = 3\pi/4$

At $x = -\pi/4 \quad y = \sin -\pi/4 + \cos -\pi/4 = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$

At $x = 3\pi/4 \quad y = 0$

4. Intercepts:

$y = 0 \quad \sin x + \cos x = 0$

$\sin x = -\cos x \quad \tan x = -1$

$x = -\pi/4$ and $x = 3\pi/4$

y – intercept : $x = 0 \quad y = 0$

4.2.9 Sketching of Rational Functions

In graphing of rational functions, we must early know the asymptotes.
Asymptotes: if the distance between the graph and some fixed line approaches zero as the graph moves farther and farther from the origin, we say that this line is asymptotes of the graph.
There are four types of asymptotes:

1. Horizontal asymptotes
2. Vertical asymptotes
3. Oblique asymptotes
4. Curved asymptotes

1. Horizontal asymptotes:

The line $y = b$ is horizontal asymptote of the graph $y = f(x)$ if either: $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

Example 19: Find the horizontal asymptotes for $y = 1/(x-1)$

Solution:

$\lim_{x\to\infty} 1/(x-1) = 1/\infty-1 = 1/\infty = 0$

$\lim_{x\to-\infty} 1/(x-1) = 1/-\infty-1 = 1/-\infty = 0$

$y = 0$ is horizontal asymptotes (x-axis)

Example 20: Find the horizontal asymptotes for the function $y = \frac{\sqrt{2x^2+1}}{3x-5}$

Solution:

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)} = \lim_{x\to-\infty} \frac{\frac{1}{-\sqrt{x^2}}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)}$$

Since $\sqrt{x^2} = f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

And we have $x \to -\infty$

Then $\sqrt{x^2} = -x \qquad x = -\sqrt{x^2} \qquad \frac{1}{x} = \frac{1}{-\sqrt{x^2}}$

$$\lim_{x\to-\infty} \frac{-\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = -\frac{\sqrt{2}}{3}$$

The horizontal asymptotes are $y = \frac{\sqrt{2}}{3}$ and $y = \frac{-\sqrt{2}}{3}$

2. Vertical asymptotes:

The line $x = a$ is vertical asymptote of the graph $y = f(x)$ if either:

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty$$

- To find vertical asymptotes, find the values of x that make the denominator equal zero and check that the limit of a function goes to (∞ or $-\infty$) as x approaches (a^+ or a^-)

Example 21: Find the vertical asymptotes for the function $y = \frac{1}{1-x}$

Solution:

$x - 1 = 0 \quad x = 1$

$\lim_{x \to 1^+} \frac{1}{x-1} = \frac{1}{0} = \infty$

$\lim_{x \to 1^-} \frac{1}{x-1} = \frac{1}{0^-} = -\infty$

$x = 1$ is vertical asymptote.

Example 22: Find the vertical asymptotes for the function $y = \frac{x^2+x-6}{x^2-4}$

Solution:

$x^2 - 4 = 0 \quad x^2 = 4 \quad x = \pm 2$

$\lim_{x \to 2^+} \frac{x^2+x-6}{x^2-4} = \frac{0}{0} \neq \infty$ or $-\infty$

$\lim_{x \to 2^-} \frac{x^2+x-6}{x^2-4} = \frac{0}{0} \neq \infty$ or $-\infty$

$x = 2$ is not a vertical asymptote.

$\lim_{x \to -2^+} \frac{x^2+x-6}{x^2-4} = \frac{-4}{0} = \infty$

$x = -2$ is the vertical asymptote.

3. Oblique (or slant) asymptotes:

When $\lim_{x \to \pm\infty} [f(x) - (mx + b)] = 0$, then the line $y = mx + b$ is oblique asymptotes for the function $f(x)$.

- To find oblique asymptotes, divide the numerator over the denominator (by long division), the result represents the oblique asymptotes.
- If the rational function has degree of numerators is one greater than the degree of denominator, the graph has an oblique asymptotes.

Example 23: Find the oblique asymptotes of $y = \frac{x^2-3}{2x-4}$

Solution: degree of numerator – degree of denominator = 2 – 1 = 1

Use long division

$y = \frac{x}{2} + 1 + \frac{1}{2x-4}$

$y = x/2 + 1$ is the oblique asymptote.

Note: A function may have oblique asymptote but is not rational function, for example $y = \sqrt{4x^2 + 9}$ has two oblique asymptotes $y = 2x$ and $y = -2x$

4. Curved asymptotes

If the degree of numerator is more than one greater than the degree of denominator, the asymptote becomes curved.

- To find curved asymptotes, use only long division.

Example 24: Find the curved asymptotes of the function $y = \frac{x^4+1}{x^2}$

Solution:

$y = x^2 + \frac{1}{x^2}$

$y = x^2$ is the curved asymptote.

Example 25: Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin.

2. *Find f and f*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x – intercept at $x = -1$,

y – intercept at ($y = 1$) at $x = 0$

$$f(x) = \frac{(1+x^2).2(x+1) - (x+1)^2.2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1$, $x = 1$

$$f(x) = \frac{(1+x^2)^2.2(-2x) - 2(1-x^2)[2(1+x^2).2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

3. *Behavior at critical points*. The critical points occur only at $x = \pm 1$ where $f(x) = 0$ (Step 2) since f exists everywhere over the domain of f. At $x = -1$, $f(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f(1) = -1 < 0$ yielding a relative maximum by the Second Derivative test.

4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f(x) < 0$ again.

5. *Inflection points*. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f is zero when $x = -\sqrt{3}$, 0 and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$ concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$ and concave up again on $(\sqrt{3}, \infty)$.

6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

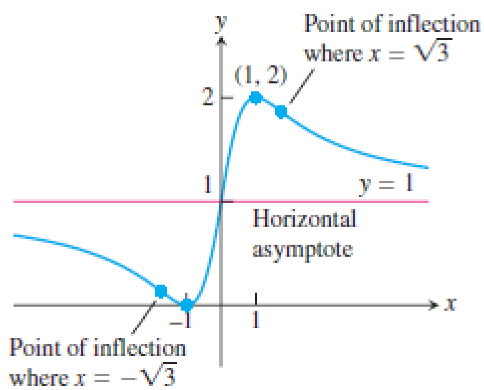
$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2}$$

$$\frac{1 + \left(\frac{2}{x}\right) + \left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right) + 1}$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$ we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$ it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$ approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $(0 \le y \le 2)$.

7. The graph of f is sketched in Figure. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$ and concave up in its approach to $y = 1$ as $x \to \infty$



Example 26: Sketch the graph of $f(x) = \frac{x^2+4}{2x}$

Solution:

1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.

2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x}$$

$$f(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2}$$

$$f(x) = \frac{4}{x^3}$$

3. The critical points occur at $x = \pm 2$ where $f(x) = 0$. Since $f(-2) < 0$ and $f(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.

4. On the interval $(-\infty, -2)$ the derivative f is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.

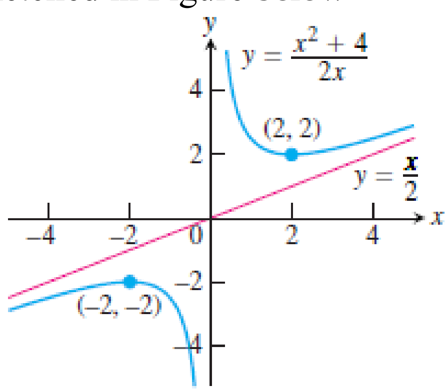
5. There are no points of inflection because $f(x) < 0$ whenever $x < 0$, $f(x) > 0$ whenever $x > 0$and f exists everywhere and is never zero throughout the domain of f. The graph is concave down on the interval $(-\infty, -2)$ and concave up on the interval $(0, \infty)$.

6. From the rewritten formula for $f(x)$, we see that

$$\lim_{x\to 0^+}\left(\frac{x}{2}+\frac{2}{x}\right) = +\infty \text{ and } \lim_{x\to 0^-}\left(\frac{x}{2}+\frac{2}{x}\right) = -\infty,$$

so the y-axis is a vertical asymptote. Also, as $x\to\infty$ or as $x\to-\infty$ the graph of $f(x)$ approaches the line $y = x/2$. Thus is an oblique asymptote.

7. The graph of f is sketched in Figure below



Example 27: Use symmetry, first derivative, second derivative, and asymptotes to graph the function $y = \frac{x^2}{x^2-1}$

Solution:

1. Symmetry:

$f(-x) = \frac{(-x)^2}{(-x)^2-1} = \frac{x^2}{x^2-1} = f(x)$

The function is even function (symmetric about y-axis)

2. First derivative:

$$y = 1 + \frac{1}{x^2 - 1}$$

$$y = 0 - \frac{2x}{(x^2 - 1)^2}$$

$$y = \frac{-2x}{(x^2-1)^2} = 2x\left(\frac{-1}{(x^2-1)^2}\right)$$

$y = 0$ $\quad \frac{-1}{(x^2-1)^2} \neq 0$

$2x = 0 \quad x = 0$

$x^2 - 1 = 0 \quad x^2 = 1 \quad x = \pm 1$ [at these values y is not defined]

at $x = 0$ $\quad y = 1 + \frac{1}{(0)^2-1} = 1 - 1 = 0$

(0, 0) is maximum point.

3. Second derivative:

$$y = \frac{-2x}{(x^2-1)^2} \qquad y = \frac{(x^2-1)^2(-2)-(-2x)[2(x^2-1)(2x)]}{(x^2-1)^4}$$

$$y = \frac{-2(x^2-1)^2+8x^2(x^2-1)}{(x^2-1)^4} = \frac{(x^2-1)[-2(x^2-1)+8x^2]}{(x^2-1)^4}$$

$$y = \frac{6x^2+2}{(x^2-1)^3} = (6x^2 + 2)\left(\frac{1}{(x^2-1)^3}\right)$$

$y = 0$ $\quad \frac{1}{(x^2-1)^3} \neq 0,$

$6x^2 + 2 = 0 \quad x^2 \neq -1/3 \quad x \neq \sqrt{-\frac{1}{3}}$

$y \neq 0$

The values of x that make y not defined:

$x^2 - 1 = 0 \quad x^2 = 1 \quad x = \pm 1$

$x = -1$, $x = 1$ are out of domain

No inflection points

4. Asymptotes:

Horizontal asymptotes:

$\lim_{x\to\infty} 1 + \frac{1}{x^2-1} = 1 + 1/\infty = 1 + 0 = 1$

$\lim_{x\to-\infty} 1 + \frac{1}{x^2-1} = 1 + 1/-\infty = 1 + 0 = 1$

$y = 1$ is horizontal asymptote.

Vertical asymptotes:

$x^2 - 1 = 0 \qquad x^2 = 1 \qquad x = \pm 1$

$\lim_{x\to1^+} 1 + \frac{1}{x^2-1} = 1 + \infty = \infty$

$\lim_{x\to-1^+} 1 + \frac{1}{x^2-1} = 1 - \infty = \infty$

$x = 1$ and $x = -1$ are vertical asymptotes.

Example 28: use first derivative, second derivative and the asymptotes to graph $y = \frac{x^2-4}{x-1}$

Solution:

1. Asymptotes:

$y = x + 1 - \frac{3}{x-1}$

$y = x + 1$ is oblique asymptote.

- Horizontal asymptote:

$\lim_{x\to\infty} \frac{x^2-4}{x-1} = \frac{\infty}{\infty}$ (indeterminate form)

No horizontal asymptote

- Vertical asymptote:

$x - 1 = 0 \qquad x = 1$

$\lim_{x\to1^+} x + 1 - 3/(x-1) = -\infty$

$x = 1$ is vertical asymptote

2. First derivative:

$y = 1 + 0 + \frac{3}{(x-1)^2}$

$= \frac{3}{(x-1)^2} + 1$

$y = 0 \qquad \frac{3}{(x-1)^2} = -1 \qquad (x-1)^2 = -3 \qquad x^2 - 2x + 4 = 0$

$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$x = \frac{-(-2)\mp\sqrt{(-2)^2-4(1)(4)}}{2(1)} = \frac{2\mp\sqrt{-12}}{2}$ (not defined)

$y \neq 0$ no max. or min. point

The value of x that makes y not defined is $x - 1 = 0 \qquad x = 1$

3. Second derivative:

$$y = 1 + \frac{3}{(x-1)^2}$$

$$y = 0 + \frac{[(x-1)^2 * 0] - [3 * 2(x-1)(1)]}{(x-1)^4}$$

$$y = \frac{-6(x-1)}{(x-1)^4} = \frac{-6}{(x-1)^3}$$

$y = 0 \qquad x = 1$ (y not defined)

$x = 1$ out of domain no inflection point

4.3 L'Hôpital's Rule

4.3.1 Indeterminate Forms 0/0

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g(x) \neq 0$ on I if $x \neq a$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

assuming that the limit on the right side of this equation exists.

Example 29: The following limits involve 0/0 indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.
Solution:

(a) $$\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{3-\cos x}{1}=\left.\frac{3-\cos x}{1}\right|_{x=0}=2$$

(b) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$$

(c) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2} \qquad \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x} \qquad \text{Still } \frac{0}{0}\text{; differentiate again.}$$

$$=\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=-\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

(d) $$\lim_{x\to 0}\frac{x-\sin x}{x^3} \qquad \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2} \qquad \text{Still } \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\sin x}{6x} \qquad \text{Still } \frac{0}{0}$$

$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Using L'Hôpital's Rule
To find

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form 0/0 at $x = a$. But as soon as one or the other of these derivatives is different

from zero at $x = a$, we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example 30: Evaluate the following limits:

a) $\lim_{x\to 0} \frac{1-cosx}{x+x^2}$, b) $\lim_{x\to 1} \frac{x^2-4x+3}{x-1}$

Solution:

a) $\lim_{x\to 0} \frac{1-cosx}{x+x^2} = \frac{0}{0}$

$$= \lim_{x\to 0} \frac{sinx}{1+2x} = \frac{0}{1} = 0$$

Note: now if we want to find $\lim_{x\to 0} \frac{sinx}{1+2x}$ by applying l'Hôpital's Rule:

$$= \lim_{x\to 0} \frac{cosx}{2} = \frac{1}{2}$$

which is **not the correct** limit. L'Hôpital's Rule can only be applied to limits that give **indeterminate forms**, and 0/1 is not an indeterminate form.

b) $\lim_{x\to 1} \frac{x^2-4x+3}{x-1}$

$$\lim_{x\to 1} \frac{x^2-4x+3}{x-1} = \frac{0}{0}$$

$$= \lim_{x\to 1} \frac{2x-4}{1} = \frac{2(1)-4}{1} = -2$$

Note: Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an indeterminant form like ∞/∞, ∞.0, or ∞ - ∞, instead of 0/0. We will consider the following form:

4.3.2 Indeterminate Forms ∞/∞

It is proved that l'Hôpital's Rule applies to the indeterminate form ∞/∞ as well as to 0/0 as shown in following examples:

Example 31: Find the limits of $\lim_{x\to \pi/2} \frac{secx}{1+tanx}$

Solution:

$$\lim_{x\to\pi/2} \frac{secx}{1+tanx} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\pi/2} \frac{secxtanx}{sec^2x}$$

$$= \lim_{x\to\pi/2} sinx = 1$$

Example 32: Find $\lim_{x\to\infty} \frac{2x^3+3x^2+1}{x^2+4}$

Solution:

$$\lim_{x\to\infty} \frac{2x^3+3x^2+1}{x^2+4} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{6x^2+6x}{2x}$$

$$= \lim_{x\to\infty} \frac{2x(3x+3)}{2x}$$

$$= \lim_{x\to\infty} 3x+3 = \infty$$

4.3.3 Indeterminate Forms ∞.0, ∞ − ∞

Sometimes these forms can be handled by using algebra to convert them to a 0/0 or ∞/∞

Example 33: Find the limits of $\lim_{x\to\infty}(x\sin\frac{1}{x})$

Solution:

$$\lim_{x\to\infty}(x\sin\frac{1}{x}) = 0.\infty$$

Let $h = 1/x$:

$$\lim_{x\to\infty}(x\sin\frac{1}{x}) = \lim_{h\to 0}\left(\frac{1}{h}\sin h\right) = \lim_{h\to\infty}\frac{sinh}{h} = 1$$

Example 34: Find the limit of this ∞ - ∞ form:

$$\lim_{x\to 0}\left(\frac{1}{sinx}-\frac{1}{x}\right)$$

Solution:

$$\lim_{x\to 0}\left(\frac{1}{sinx}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{1}{sinx}-\lim_{x\to 0}\frac{1}{x} = \infty-\infty$$

$$\lim_{x\to 0}\left(\frac{1}{sinx}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-sinx}{xsinx}$$

$$=\lim_{x\to 0}\frac{1-cosx}{sinx+xcosx}$$

Still 0/0

Use L'Hôpital's Rule again:

$$=\lim_{x\to 0}\frac{sinx}{2cosx-xsinx}=\frac{0}{2}=0$$

4.4 Applied Optimization

In this section we use derivatives to solve a variety of optimization problems in business, mathematics, physics, and economics.

Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.

4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Example 35: An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution:

We start with a picture (Figure 23).

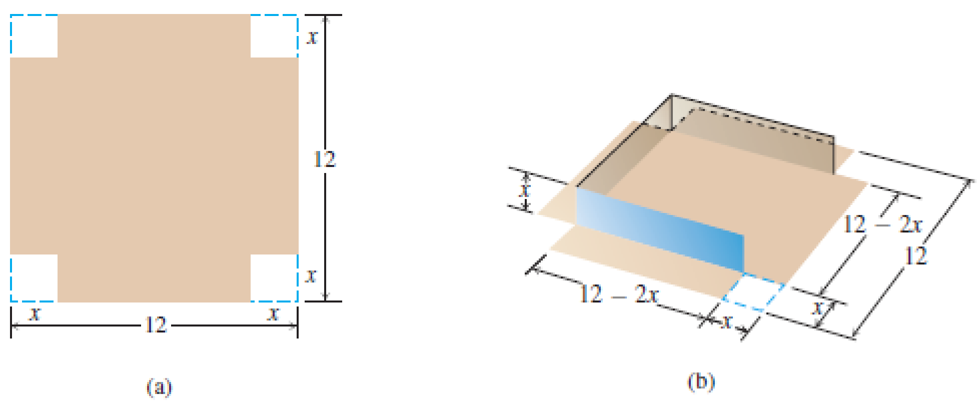


Figure 23

In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3$$

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Figure 24) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$.To learn more, we examine the first derivative of V with respect to x:

$$dV/dx = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x)$$

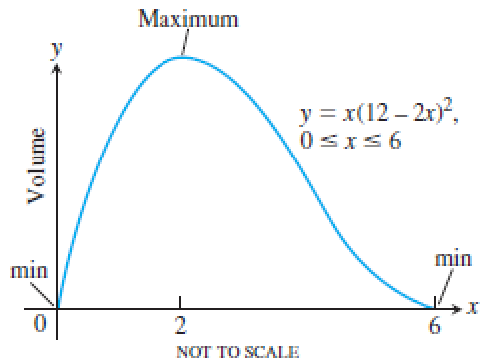


Figure 24

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are:

Critical-point value: $V(2) = 128$
Endpoint values: $V(0) = 0$, $V(6) = 0$.

The maximum volume is 128 in^3. The cutout squares should be 2 in. on a side.

Example 36: You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 25). What dimensions will use the least material?
Solution:

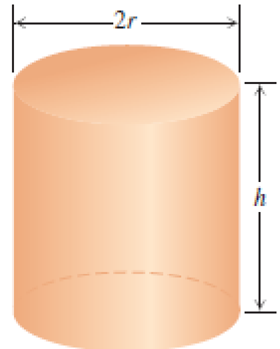


Figure 25

Volume of can: If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000 \qquad [1 \text{ liter} = 1000 \text{ cm}^3]$$

Surface area of can:

$$A = \underbrace{2\pi r^2}_{\substack{\text{circular}\\ \text{end}}} + \underbrace{2\pi r h}_{\substack{\text{cylindrical}\\ \text{wall}}}$$

How can we interpret the phrase "least material"? For a first approximation we can ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$.

To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier:

$$h = \frac{1000}{\pi r^2}$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \left(\frac{2000}{r}\right) \end{aligned}$$

Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 26 suggests that such a value exists.

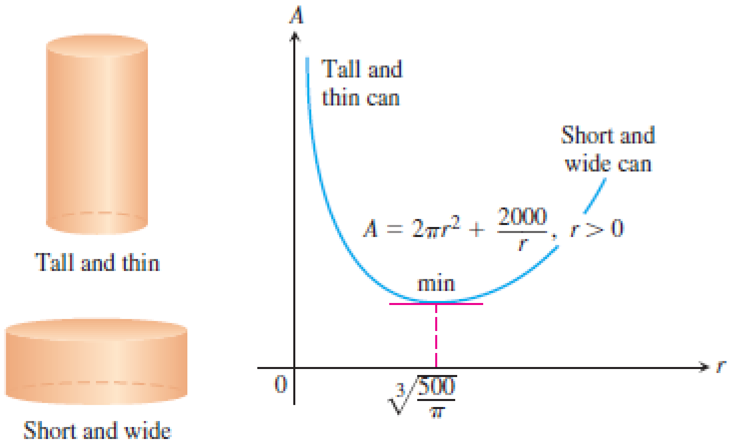


Figure 26

Notice from the graph that for small r (a tall, thin cylindrical container), the term $2000/r$ dominates and A is large. For large r (a short, wide cylindrical container), the term $2\pi r^2$ dominates and A again is large.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$
$$0 = 4\pi r - 2000/r^2$$
$$4\pi r^3 = 2000$$
$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore everywhere concave up and the value of A at $r = \sqrt[3]{500/\pi}$ is an absolute minimum.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

The one-liter can that uses the least material has height equal to twice the radius, here with $r \approx 5.42$ cm and $h \approx 10.84$ cm.

Example 37: Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$.

Solution:

$A = 2x\,(12 - x^2)$

$A = 24x - 2x^3 \qquad 0 \le x \le 2\sqrt{3}$

$dA/dx = 24 - 6x^2$

At maximum or minimum points, $dA/dx = 0$

$24 - 6x^2 = 0$

$x^2 = 24/6 = 4$

$x = 2$ or $x = -2$ [$x = -2$ is neglected]

$d^2A/dx^2 = -12x$

At $x = 2 \qquad d^2A/dx^2 = -12 * 2 = -24 = -ve$

$x = 2$ is maximum point

Check bound:

At $x = 0 \qquad A = 0$

At $x = 2\sqrt{3} \qquad A = 0$

At $x = 2$ Absolute max.

$A = 2(2)\,[12 - (2)^2] = 32$ unit

Example 38: The height of an object moving vertically is given by $S = -16t^2 + 96t + 112$ when s in feet and t in seconds. Find:

a. The velocity when $t = 0$
b. Its maximum height
c. Its velocity when $s = 0$

Solution:

a. $Velocity = ds/dt = -32t + 96$

At $t = 0$ $\quad v = -32(0) + 96 = 96$ ft/sec

b. At maximum height, velocity $v = 0$

$-32t + 96 = 0$

$t = 96/32 = 3$ sec

$S_{max} = -16(3)^2 + 96(3) + 112 = 256$ ft

c. At $S = 0$ $\quad -16t^2 + 96t + 112 = 0$ $\quad -t^2 + 6t + 7 = 0$

$t^2 - 6t - 7 = 0$

$(t - 7)(t + 1) = 0$

$t = 7$

$t = -1$ (neglected)

$v = -32t + 96 = -32(7) + 96 = -128$ ft/sec

Example 39: what is the smallest perimeter possible for a rectangle of area equal to 16 cm^2.

Solution:

$P = 2(x + y)$

$A = xy$

$16 = xy \qquad y = 16/x$

$P = 2(x + 16/x) = 2x + 32/x \qquad 0 < x < \infty$

$$\frac{dp}{dx} = 2 - \frac{32}{x^2} = \frac{2x^2 - 32}{x^2}$$

$\frac{dp}{dx} = 0 \qquad \frac{2x^2-32}{x^2} = 0 \qquad 2x^2 - 32 = 0 \qquad x^2 = 16 \qquad x = \pm 4$

$x = 4$ [$x = -4$ was neglect]

$$\frac{d^2p}{dx^2} = 0 - \frac{-32(2x)}{x^4} = \frac{64}{x^3}$$

At $x = 4$ $\qquad \frac{d^2p}{dx^2} = +ve$

$x = 4$ local min. point

Bound check:

At $x = 0 \qquad P = \infty$

At $x = \infty \qquad P = \infty$

$x = 4$ give absolute min., $\quad y = 16/4 = 4$

$P = 2(4 + 4) = 16$ cm (rectangle is square)

Example 40: The wall shown is 8ft height and stands 27 ft from the building. What is the length of the shortest straight beam that will reach to the side of building from the ground outside the wall?

Solution:

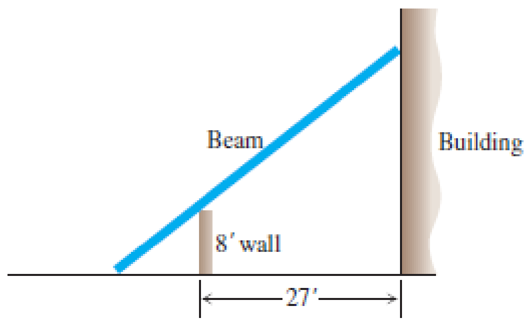


Figure 27

Let L = length of the beam

$L^2 = y^2 + (x + 27)^2$

From similar triangle:

$\frac{y}{27+x} = \frac{8}{x}$ $\quad$ $y\,x = 8(27 + x)$ $\quad$ $y = \frac{8(27+x)}{x}$

$L^2 = [\frac{8(27+x)}{x}]^2 + (x+27)^2$

$L = [(\frac{8(27+x)}{x})^2 + (x+27)^2]^{1/2}$ $\quad$ $0 < x < \infty$

For min. L: $dL/dx = 0$

$$\frac{dL}{dx} = \frac{1}{2}[(\frac{8(27+x)}{x})^2 + (x+27)^2]^{-\frac{1}{2}} \times [2\left(\frac{8(27+x)}{x}\right) * \frac{8x-216-8x}{x^2} + 2(x+27) * 1]$$

$$\frac{dL}{dx} = \frac{2[\frac{216+8x}{x} * \left(\frac{-216}{x^2}\right) + (x+27)]}{2[\left(\frac{8(27+x)}{x}\right)^2 + (x+27)^2]^{1/2}}$$

$$\frac{dL}{dx} = \frac{\frac{-46656 - 1728x}{x^3} + \frac{(x+27)}{1}}{2[\left(\frac{8(27+x)}{x}\right)^2 + (x+27)^2]^{1/2}}$$

$$\frac{dL}{dx} = \frac{\frac{-46656 - 1728x + x^4 + 27x^3}{x^3}}{2[\left(\frac{8(27+x)}{x}\right)^2 + (x+27)^2]^{1/2}}$$

$$\frac{dL}{dx} = 0$$

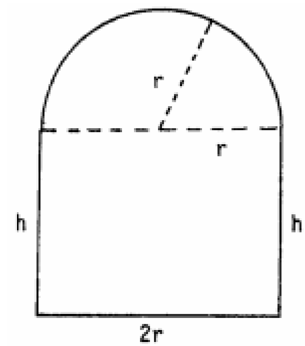
$$\frac{-46656 - 1728x + x^4 + 27x^3}{x^3} = 0$$

$$x^4 + 27x^3 - 1728x - 46656 = 0$$

By trail an error $x = 12$ ft
At $x = 12$ local min.

Bound check:
At $x = 0$ $L = \infty$
At $x = \infty$ $L = \infty$
$x = 12$ ft give absolute minimum of L

Example 41: A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.
Solution:
From the diagram:



The perimeter is: $P = 2r + 2h + \pi r$
where,
r = radius of semicircle
h = the height of the triangle
The amount of light transmitted proportional to:
$A = 2rh + (1/4)\pi r^2$
$A = r(P - 2r - \pi r) + (1/4)\pi r^2$
$= rP - 2r^2 - (3/4)\pi r^2$
$dA/dr = P - 4r - (3/2)\pi r$

$P - 4r - (3/2)\pi r = 0$
$r = 2P/(8 + 3\pi)$
$2h = P - 4P/(8 + 3\pi) - 2\pi P/(8 + 3\pi)$
$= (4 + \pi)P/(8 + 3\pi)$
Therefore, $2r/h = 8/(4 + \pi)$ gives the proportions that admit the most light since $d^2A/dr^2 = -4 - 3\pi/2 < 0$