# CHAPTER 7

# TRANSCENDENTAL FUNCTIONS

A function that is not algebraic (cannot expressed in terms of algebra) is called transcendental function.

The transcendental functions are:

1. Trigonometric functions
2. Inverse trigonometric functions
3. Logarithmic functions
4. Exponential functions

## 7.1 Inverse Functions

A function that undoes, or inverts, the effect of a function $f$ is called the *inverse* of $f$.

### One-to-One Functions

A function is a rule that assigns a value from its range to each element in its domain. (i.e for each value of $x$, there is only one value of $y$)

For example: $y = x^3$ one-to-one function

$y = 4x - 2$ one-to-one function

$y = x^2$ not one-to-one function

Some functions assign the same range value to more than one element in the domain.

The symbol for the inverse function is $f^{-1}$

$x \rightarrow f \rightarrow y \rightarrow f^{-1} \rightarrow x = f^{-1}(f(x))$

- Only one-to-one function have inverse.

### Finding Inverses

To find the inverse of a function $f(x)$:

1. Express $x$ in terms of $y$ ($x = f(y)$)

2. Interchange $x$ and $y$ in the formula of step 1, we get the function $g(x)$ which is the inverse of $f(x)$.
3. Checking the inverse function by finding $f(g(x))$ and $g(f(x))$, if $f(g(x))= g(f(x))=x$, the $f(x)$ and $g(x)$ are inverses of one another.

**Example 1:** Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of $x$.

**Solution:**

1. *Solve for x in terms of y*:

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

2. *Interchange x and y*:

$$y = 2x - 2$$

The inverse of the function $y = \frac{1}{2}x + 1$ is the function $f^{-1}(x) = 2x - 2$. (Figure 1)

3. To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x$$

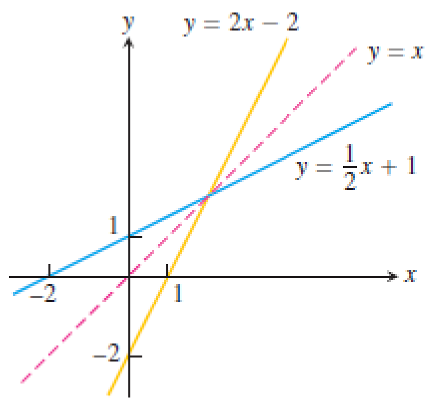



Figure 1

**Example 2**: Find the inverse of $y = x/4 + 3$

**Solution**: $y = \frac{x+12}{4}$, $y$ is one-to-one function

1. Find $x = f(y)$
   $4y = x + 12 \qquad x = 4y - 12 = f(y)$

2. $y = 4x - 12$
3. Check: $f(x) = \frac{x+12}{4}$, $g(x) = 4x - 12$

$$f(g(x) = \frac{(4x-12)+12}{4} = x$$

$$g(f(x) = 4\left(\frac{x+12}{4}\right) - 12 = x = f(g(x)$$

$g(x)$ is the inverse of $f(x)$

### 7.2 Derivatives of Inverse Functions

We calculated the inverse of the function $f(x) = \frac{1}{2}x + 1$ as $f^{-1}(x) = \frac{1}{2}x + 1$ in Example 1. If we calculate their derivatives, we see that:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2$$

**The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line.**

**Example 3:** Let $f(x) = x^3 - 2$. Find the value of $df^{-1}/dx$ at $x = 2$ without finding a formula for $f^{-1}(x)$.

**Solution**:

$df/dx\,\big|_{x=2} = 3x^2\,\big|_{x=2} = 12$

$df^{-1}/dx\,\big|_{x=2} = 1/\left(df/dx\,\big|_{x=2}\right) = 1/12$

see Figure 2

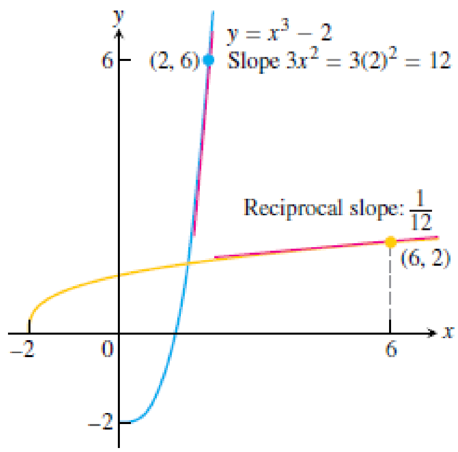



Figure 2

### 7.3 Logarithmic Functions

If $a$ is any positive real number other than 1, the base $a$ exponential function $f(x)=a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the *logarithm function with base a*.

**DEFINITION:** The **logarithm function with base** $\boldsymbol{a}$, $y=\log_a x$, is the inverse of the base $a$ exponential function $y=a^x$ ($a > 0$, $a \neq 1$).

The domain of $\log_a x$ is $(0, \infty)$ = the range of $a^x$.
The range of $\log_a x$ is $(-\infty, \infty)$ = the domain of $a^x$.
Figure 3 shows the graphs of four logarithmic functions with $a > 1$.

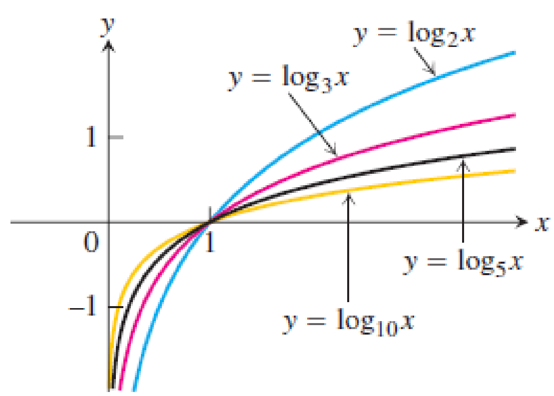


Figure 3

Figure 4 shows the graph of $y = \log_2 x$.
The graph of $y = a^x$ , $a > 1$, increases rapidly for $x > 0$, so its inverse, $y = \log_a x$, increases slowly for $x > 1$.

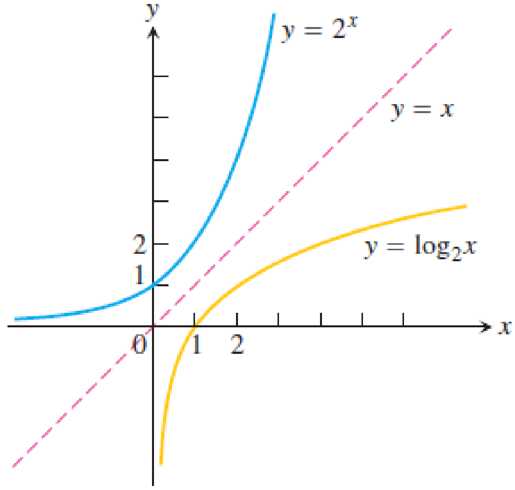



Figure 4

We can obtain the graph of $y = \log_a x$ by reflecting the graph of the exponential $y = a^x$ across the line $y = x$.
Logarithms with base 2 are commonly used in computer science. Logarithms with base $e$ and base 10 are so important in applications that calculators have special keys for them. They also have their own special notation and names (Figure 5):

$\log_e x$ is written as $\ln x$.
$\log_{10} x$ is written as $\log x$.

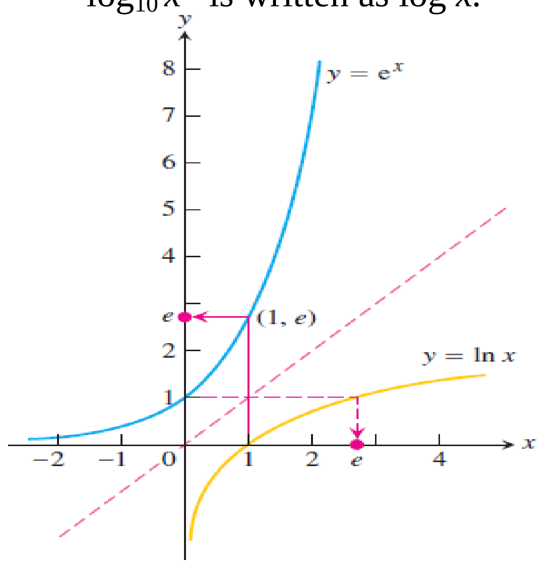


Figure 5

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**. For the natural logarithm,

$$\boxed{\ln x = y \leftrightarrow e^y = x}$$

In particular, if we set $x = e$, we obtain

$$\boxed{\ln e = 1}$$

Because $e^1 = e$
$e \approx 2.71828$

**7.4 Algebraic Properties of the Natural Logarithm**
For any numbers b > 0 and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln b/x = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln 1/x = -\ln x$ (Rule 2 with $b = 1$)
4. *Power Rule*: $\ln x^r = r \ln x$

**Example 4**: Here are examples of the natural logarithm properties:

a. $\ln 4 + \ln \sin x = \ln (4 \sin x)$
b. $\ln \frac{x+1}{2x-3} = \ln(x + 1) - \ln(2x - 3)$
c. $\ln \frac{1}{8} = -\ln 8$

$= -\ln 2^3 = -3 \ln 2$

Because $a^x$ and $\log_a x$ are inverses, composing them in either order gives the identity function:

**Properties for $a^x$ and $\log_a x$:**

**1.** Base $a$: $a^{\log_a x} = x$, $\log_a a^x = x$ $\qquad a > 0,\ a \neq 1,\ x > 0$

**2.** Base $e$: $e^{\ln x} = x$, $\ln e^x = x$ $\qquad x > 0$

**3**. $a^x = e^{x\ln a}$

4. $\log_a x = \dfrac{\ln x}{\ln a}$

To explain:

1. $a^x = e^{\ln(a^x)}$

$$= e^{x\ln a}$$

$$= e^{(\ln a)x}$$

2. $2^x = e^{(\ln 2)x} = e^{x\ln 2}$

3. $5^{-3x} = e^{(\ln 5)(-3x)} = e^{-3x\ln 5}$

## 7.5 Definition of the Natural Logarithm Function (ln *x*)

The natural logarithm of a positive number $x$, written as **ln** $x$, is the value of an integral:

$$\ln x = \int_{1}^{x} \frac{1}{t}\,dt, \qquad x > 0$$

= area under the curve $y = 1/t$ and bounded by lines $t = 1$ and $t = x$(Figure 6)

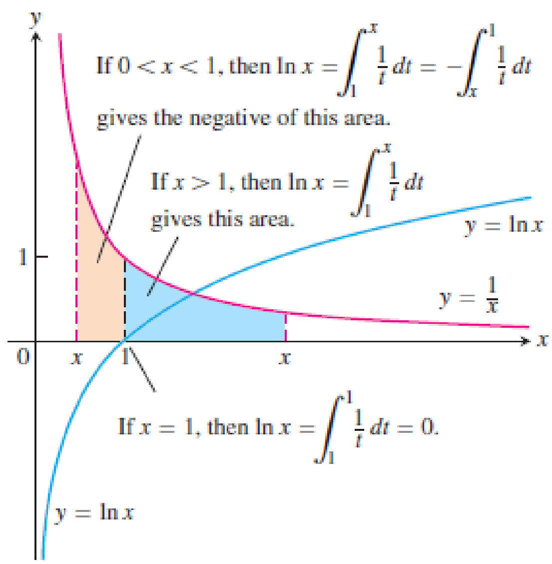



Figure 6

If $x = 1$ $\qquad \ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$

For $x > 1$ $\quad \ln x$ is $+\,ve$

For $0 < x < 1$ $\quad \ln x$ is $-\,ve$

$\text{Lim}_{x\to\infty} \ln x = \infty,$

$\text{Lim}_{x\to 0^+} \ln x = -\infty$

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

If $y = \ln u$, $u = f(x)$

$$\frac{d}{dx}\ln u = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\cdot\frac{du}{dx}$$

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}} \qquad \boxed{\int \frac{1}{u}\,du = \ln|u| + c}$$

**Example 5**: Find *dy/dx* for the function $y = x^2 \ln(4x)$

**Solution**: $\frac{dy}{dx} = x^2\left(\frac{1}{4x}\cdot 4\right) + \ln(4x).\,2x$

$= x + 2x\ln(4x) = x\,(1 + 2\ln(4x))$

**Example 6**: $y = \ln(\tan x + \sec x)$, find *dy/dx*

**Solution**:

$$\frac{dy}{dx} = \frac{1}{tanx + \sec x}\left[sec^2x + secx\ tanx\right]$$

$$= \frac{secx(secx + tanx)}{secx + tanx} = secx$$

$$\therefore \int secx\,dx = \ln|secx + tanx| + C$$

**Example 7**: Evaluate $\int \frac{x}{x^2+3}\,dx$

**Solution**: let $u = x^2 + 3$

$du = 2x\,dx$

$x\,dx = du/2$

$$\int \frac{1}{u}\cdot\frac{du}{2} = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+3| + C$$

**Example 8**: Evaluate $\int \frac{x}{x+1}\,dx$

**Solution**: $\frac{x}{x+1} = 1 - \frac{1}{x+1}$

$$\int \frac{x}{x+1}\,dx = \int\left(1 - \frac{1}{x+1}\right)dx$$

$$= \int dx - \int\frac{dx}{x+1} = x - \ln|x+1| + C$$

**Note**: sometimes we need ($\ln x$) to find the derivative of functions that involve products, quotient and powers quickly.

**Example 9**: If $y = \left(\sqrt{x+3}\right)(sinx\ cosx)$, find $dy/dx$

**Solution**: $\ln y = \ln\left(\sqrt{x+3}\right)(sinx\ cosx)$

$\ln y = \ln\sqrt{x+3} + \ln(sinx\ cosx) = \ln\,(x+3)^{1/2} + \ln\sin x + \ln\cos x$

$\ln y = \frac{1}{2}\ln\,(x+3) + \ln \sin x + \ln \cos x$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x+3}\right) + \frac{1}{\sin x}(\cos x) + \frac{1}{\cos x}(-\sin x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2(x+3)} + \cot x - \tan x$$

$$\frac{dy}{dx} = y[\frac{1}{2(x+3)} + \cot x - \tan x]$$

**Example 10**: find *dy/dx* for $y^5 = \sqrt{\frac{(x+1)^5}{(x+2)^{10}}}$

**Solution**:

$$y^5 = \left(\frac{(x+1)^5}{(x+2)^{10}}\right)^{1/2} = \frac{(x+1)^{5/2}}{(x+2)^5}$$

$$\ln y^5 = \ln \frac{(x+1)^{5/2}}{(x+2)^5}$$

$$5 \ln y = \ln(x+1)^{5/2} - \ln(x+2)^5$$

$$5 \ln y = \frac{5}{2}\ln(x+1) \quad - 5\ln(x+2)$$

$$\ln y = \frac{1}{2}\ln(x+1) \quad - \ln(x+2)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x+1}\right) - (\frac{1}{x+2)} = \frac{1}{2(x+1)} - \frac{1}{(x+2)} = \frac{x+2-2(x+1)}{2(x+1)(x+2)}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{-x}{2(x+1)(x+2)}$$

$$\frac{dy}{dx} = \frac{-xy}{2(x+1)(x+2)}$$

## 7.6 The Integrals of tan *x*, cot *x*, sec *x*, and csc *x*

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u}$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C$$

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{du}{u}$$

$$= \ln|u| + C = \ln|\sin x| + C$$

$$= -\ln|\csc x| + C$$

$$\int \sec x\, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

$$\int \csc x\, dx = \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)} dx = \int \frac{csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$$

## 7.7 The Inverse of ln *x* and the Number *e*

The function ln $x$, being an increasing function of $x$ with domain $(0, \infty)$ and range $(-\infty, \infty)$ has an inverse $\ln^{-1}x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1}x$ is the graph of ln $x$ reflected across the line $y = x$, as you can see in Figure 7.

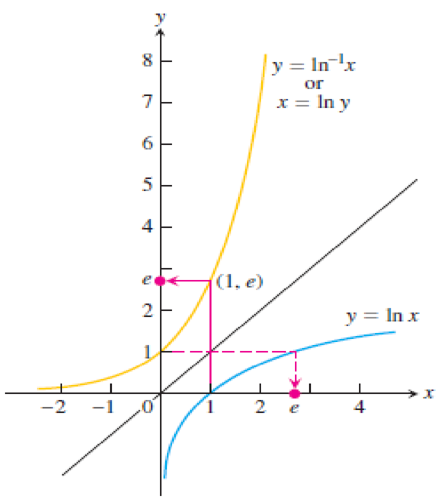



Figure 7

The function $y = \ln^{-1}x$ is also denoted by $\exp x$

The function $y = \exp x$ is the inverse of $y = \ln x$

$\ln(\exp x) = x$, for all $x$

$\exp(\ln(x)) = x$, for $x > 0$

The number $e$ was defined to satisfy the equation $\ln(e) = 1$, so $e = \exp(1)$.

We can show that $\ln^{-1}x = \exp x$ is an exponential function with base $e$:

For $e^x$ $\quad \ln e^x = x \ln e = x(1) = x$

$\ln e^x = 1$ and $\ln(\exp(x)) = x$

$\exp(x) = e^x$

**Laws of Exponents for $e^x$**

For all numbers $x$, $x_1$ and $x_2$, the natural exponential $e^x$ obeys the following laws:

1. $e^{x1} \cdot e^{x2} = e^{x1+x2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x1}}{e^{x2}} = e^{x1-x2}$
4. $(e^{x1})^{x2} = e^{x1x2} = (e^{x2})^{x1}$

**Notes**: 1. To remove logarithms from an equation, exponentiate both sides.

2. To remove exponents from an equation, take logarithm from both sides.

**Example 11**: Find $y$ for the following equations:

1. $\ln y = x^2$
2. $\ln (y - 2) = \ln (\sin x) - x$

**Solution**:

1. $\ln y = x^2$
   $e^{\ln y} = e^{x2}$
   $y = e^{x2}$

2. $\ln (y - 2) = \ln (\sin x) - x$
   $\ln (y - 2) - \ln (\sin x) = x$
   $\ln \dfrac{y-2}{\sin x} = -x$
   $e^{\ln \frac{y-2}{\sin x}} = e^{-x}$
   $\dfrac{y-2}{\sin x} = e^{-x}$
   $y = e^{-x} \sin x + 2$

## 7.8 The Derivative and Integral of $e^x$

$y = e^x$

$\ln y = \ln e^x$

$\ln y = x$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y$$

$\therefore \frac{d}{dx} e^x = e^x$

$$\boxed{\frac{d}{dx} e^u = e^u \frac{du}{dx}}$$

The integration of $e^x$:

$$\boxed{\int e^u\,du = e^u + C}$$

**Example 12**: Find $dy/dx$ for the function $y = e^{\tan x}$

**Solution**: $\frac{d}{dx} e^{\tan x} = e^{\tan x} \cdot sec^2 x$

**Example 13**: Evaluate $\int_{-\ln x(a+1)}^{0} e^{-x}\,dx$

**Solution**: $\int_{-\ln x(a+1)}^{0} e^{-x} = -\int_{-\ln x(a+1)}^{0} e^{-x}\ (-dx) = [-e^{-x}]_{-\ln x(a+1)}^{0}$

$$= -\left[e^0 - e^{-(-\ln x(a+1))}\right] = -[1-(a+1)] = +a$$

**Example 14**: Evaluate $\int_0^{\ln 2} \frac{24}{e^{-3x}}\,dx$

**Solution**: $\int_0^{\ln 2} \frac{24}{e^{-3x}}\,dx = 24\int_0^{\ln 2} e^{3x}dx = \frac{24}{3}\int_0^{\ln 2} e^{3x} \cdot 3dx$

$$= 8[e^{3x}]_0^{\ln 2} = 8\left[e^{3\ln 2} - e^0\right] = 8[e^{\ln 2^3} - 1]$$

$= 8\ [2^3 – 1] = 64 – 8 = 56$

**Example 15**: Evaluate $\int_e^{e^2} \frac{dx}{x\ \ln x}$

**Solution**: let $u = \ln x \rightarrow du = (1/x)\ dx$

*Upper Limit* = $\ln e^2 = 2 \ln e = 2$

Lower limit = $\ln e = 1$

$$\int_1^2 \frac{du}{u} = [\ln|u|]_1^2$$

$= \ln 2 – \ln 1 = \ln 2 – 0 = \ln 2$

**Example 16**: If $y = x^3 e^{-2x} \cos 3x$, find $dy/dx$

**Solution**: $y = x^3 e^{-2x} \cos 3x$

$\ln y = \ln (x^3 e^{-2x} \cos 3x)$

$\ln y = \ln (x^3 e^{-2x} \cos 3x)$

$\ln y = \ln x^3 + \ln e^{-2x} + \ln \cos 3x$

$\ln y = 3 \ln x - 2x \ln e + \ln \cos 3x$

$\ln y = 3 \ln x - 2x + \ln \cos 3x$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 3\frac{1}{x} - 2 + \frac{1}{\cos 3x}(-3 \sin 3x)$$

$$\frac{dy}{dx} = \left(\frac{3}{x} - 2 - 3 \tan 3x\right) . y$$

$$= \left(\frac{3}{x} - 2 - 3 \tan 3x\right) x^3 e^{-2x} \cos 3x$$

**Example 17**: Find the maximum value of $f(x) = x^2 \ln \frac{1}{x}$

**Solution**: $y = x^2 \ln \frac{1}{x} = x^2 \ln x^{-1} = -x^2 \ln x$

$$\frac{dy}{dx} = -x^2 \left(\frac{1}{x}\right) + \ln x\,(-2x) = -x - 2x \ln x = -x(1 + 2 \ln x)$$

To find maximum value:

$dy/dx = 0 \rightarrow -x\,(1 + 2 \ln x) = 0$

$-x = 0 \rightarrow x = 0$

$1 + 2 \ln x = 0 \rightarrow \ln x = -(1/2)$

$e^{\ln x} = e^{-½} \rightarrow x = e^{-½}$

For $x = 0$, the function is not defined $[0 * \ln 1/0 = 0 * \infty]$

For $x = e^{-½} = \frac{1}{\sqrt{e}}$

At $x = \frac{1}{\sqrt{e}}$, the function has local maximum value

At $x = \frac{1}{\sqrt{e}} \to y = -\left(\frac{1}{\sqrt{e}}\right)^2 \cdot \ln e^{-\frac{1}{2}} = -\frac{1}{e} \cdot -\frac{1}{2} \ln e = \frac{1}{2e}$

At the domain of *f(x)* is $x > 0$ and at $x > \frac{1}{\sqrt{e}}$, the function decreasing, then $y = 1/2e$ is absolute maximum value.

## 7.9 The General Exponential Function $a^x$

Since $a = e^{\ln a}$ for any positive number $a$, we can think of $a^x$ *as* $(e^{\ln a})^x = e^{x\ln a}$. We therefore make the following definition:
For any numbers $a > 0$ and $x$, the exponential function with base $a$ is given by

$$\boldsymbol{a^x = e^{x\ln a}}$$

**Laws of exponents:**

1. $a^x + a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $a^{-x} = \frac{1}{a^x}$
4. $(a^x)^y = a^{xy} = (a^y)^x$

**Derivative of $y = a^x$**

$y = a^x$

$\ln y = \ln a^x$

$\ln y = x \ln a$

$(1/y)\,(dy/dx) = \ln a$

$(dy/dx) = y \ln a = a^x \ln a$

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}$$

The integral equivalent of this last result is

$$\int a^u du = \frac{a^u}{\ln a} + C$$

## 7.10 Logarithms with Base *a*

If $a$ is any positive real number other than 1, the function $a^x$ is one-to-one and has a nonzero derivative at every pont. It therefore has an inverse. Its inverse is called the *logarithm function with base a*.

For any positive number a $\neq$ 1, the **logarithm of $x$ with base $a$**, denoted by **$\log_a x$**, is the inverse function of $a^x$.

The graph of $\log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the 45° line (Figure 8).

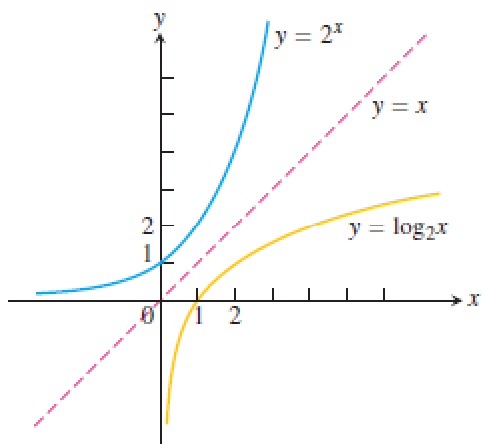



Figure 8

When $a = e$, we have $\log_e x$ = inverse of $e^x = \ln x$. Since $\log_a x$ and $a^x$ are inverses of one another, composing them in either order gives the identity function.

**Properties for $a^x$ and $\log_a x$:**

**1.** Base $a$: $a^{\log_a x} = x$, $\log_a a^x = x$ $\qquad a > 0,\ a \neq 1,\ x > 0$

**2.** Base $e$: $e^{\ln x} = x$, $\ln e^x = x$ $\qquad x > 0$

**3**. $a^x = e^{x\ln a}$

4. $\log_a x = \dfrac{\ln x}{\ln a}$

### Rules for base *a* logarithms

For any numbers x > 0 and y > 0,

1. *Product Rule*:
   $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule*:
   $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule*:
   $\log_a \dfrac{1}{y} = -\log_a y$
4. *Power Rule*:
   $\log_a x^y = y \log_a x$

### 7.11 Derivatives and Integrals Involving $\log_a x$

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}$$

$$\boldsymbol{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}}$$

**Example 18**:

**a.** $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$

**b.** $\int \frac{\log_2 x}{x}dx = \frac{1}{\ln 2}\int\frac{\ln x}{x}dx \qquad (\log_2 x = \frac{\ln x}{\ln 2})$

$$= \frac{1}{\ln 2}\int u\,du \qquad (u = \ln x,\ du = (1/x)\,dx)$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

**Example 19**: Evaluate $\int_{1}^{\sqrt{2}} 2^{x^2} x\, dx$

**Solution**:

$$\int_{1}^{\sqrt{2}} 2^{x^2} \cdot x\, dx = \frac{1}{2}\left[\frac{2x^2}{\ln 2}\right]_{1}^{\sqrt{2}}$$

$$= \frac{1}{2}\left[\frac{2^2}{\ln 2} - \frac{2^1}{\ln 2}\right] = \frac{1}{2}\left[\frac{2}{\ln 2}\right] = \frac{1}{\ln 2}$$

**Example 20**: Find *dy/dx* for $y = \frac{1}{\log_2 x}$

**Solution:**

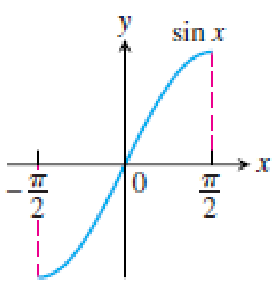
$$y = (\log_2 x)^{-1} \rightarrow \frac{dy}{dx} = -(\log_2 x)^{-2}\left(\frac{1}{x \ln 2}\right) = \frac{-1}{x \ln 2 (\log_2 x)^2}$$

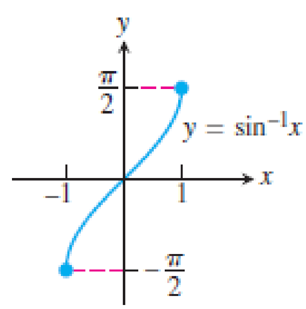
## 7.12 Inverse Trigonometric Functions

The inverse trigonometric Functions are used to find the angles from the triangle sides. They also provide antiderivatives for a wide variety of functions.
The six basic trigonometric functions of a general radian angle $x$ were reviewed. These functions are not one-to-one (their values repeat periodically).
However, we can restrict their domains to intervals on which they are one-to-one.

1. ***y = f(x) =*** $\boldsymbol{\sin^{-1} x}$

$y = \sin x$ $(-\infty < x < \infty)$ is not one-to-one
The sine function increases from -1 at $x = -\pi/2$ to +1 at $x = \pi/2$.
By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse called $\sin^{-1}x$.
$y = \sin^{-1}x$ (arc sin $x$) $\leftrightarrow$ $x = \sin y$
$\sin^{-1}(\sin x) = x$, $\sin(\sin^{-1} x) = x$
**Note**: $\sin^{-1} x \neq (\sin x)^{-1}$

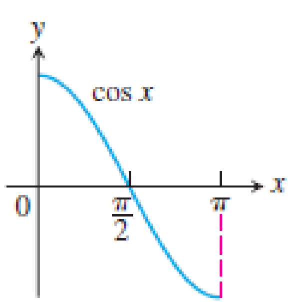


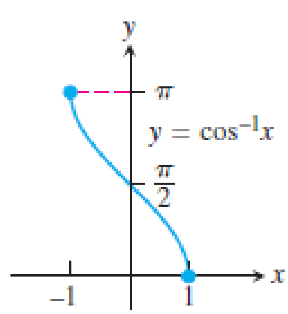




Domain: $-\pi/2 \leq x \leq \pi/2$
Range: $-1 \leq y \leq 1$

Domain: $-1 \leq x \leq 1$
Range: $-\pi/2 \leq y \leq \pi/2$

2. ***y = f(x) =*** $\boldsymbol{\cos^{-1} x}$

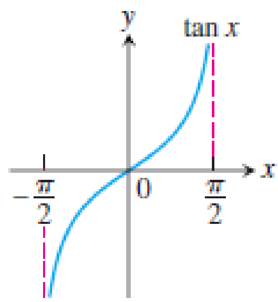


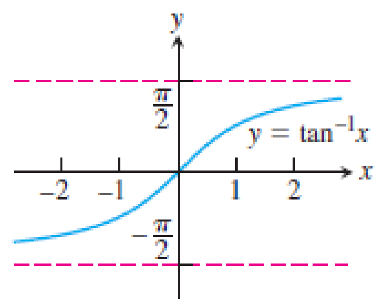



Domain: $0 \le x \le \pi/2$
Range: $-1 \le y \le 1$

Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi/2$

**3.** $\boldsymbol{y = f(x) = \tan^{-1} x}$

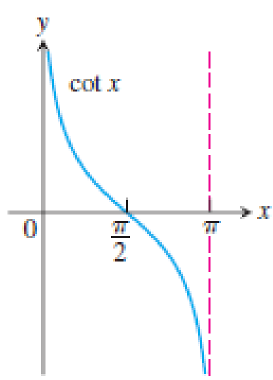


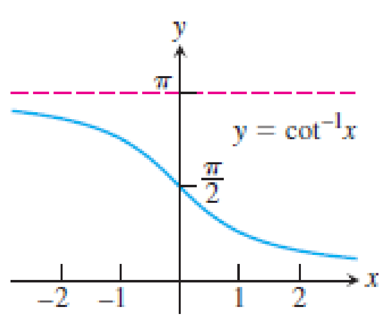




Domain: $-\pi/2 \le x \le \pi/2$
Range: $-\infty < y < \infty$

Domain: $-\infty < x < \infty$
Range: $-\pi/2 \le yx \le \pi/2$

**4.** $\boldsymbol{y = f(x) = \cot^{-1} x}$

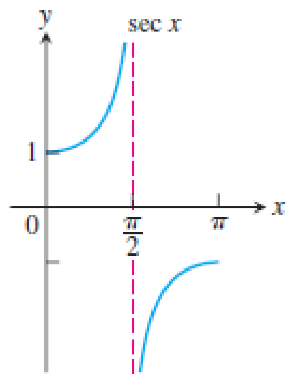


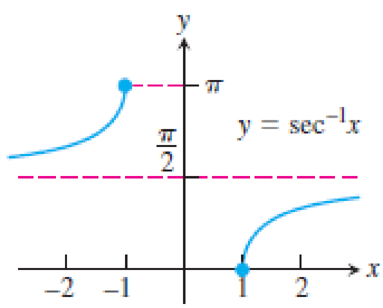




Domain: $0 \le x \le \pi/2$
Range: $-\infty < y < \infty$

Domain: $-\infty < x < \infty$
Range: $0 \le y \le \pi$

**5.** $y = f(x) = sec^{-1} x$

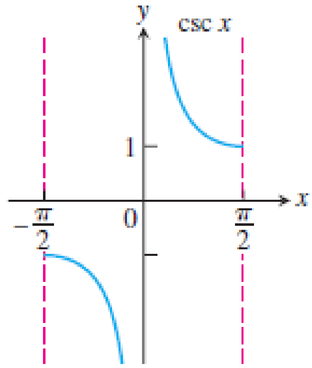


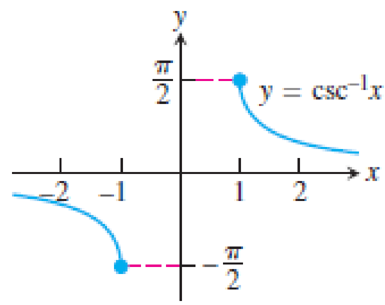




Domain: $0 \le x \le \pi$ ($x \ne \pi/2$)
Range: $-\infty < y \le -1,\ 1 < y < \infty$

Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi$ ($y \ne \pi/2$)

**6.** $y = f(x) = csc^{-1} x$



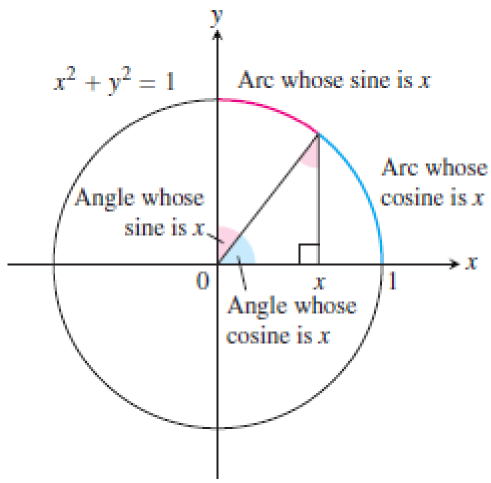




Domain: $-\pi/2 \le x \le \pi/2$ ($x \ne 0$)
Range: $-\infty < y \le -1,\ 1 \le y < \infty$

Domain: $x \le -1$ or $x \ge 1$
Range: $-\pi/2 \le y \le \pi/2$ ($y \ne 0$)

These restricted functions are now one-to-one, they have inverses, which we denote by:

$y = \sin^{-1} x$ or $y = \arcsin x$
$y = \cos^{-1} x$ or $y = \arccos x$
$y = \tan^{-1} x$ or $y = \arctan x$
$y = \cot^{-1} x$ or $y = \text{arccot}\, x$
$y = \sec^{-1} x$ or $y = \text{arcsec}\, x$
$y = \csc^{-1} x$ or $y = \text{arccsc}\, x$




**Example 21:** Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution:

(a) we see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Because $\sin\frac{\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)$ and π/3 belongs to the range [-π/2, π/2] of the arcsine function. See Figure 9

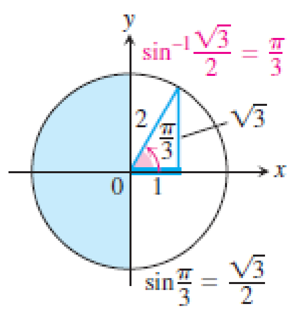



Figure 9

(b) We have $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Because cos (2π/3) = - ½ and 2π/3 belongs to the range [0, π] of the arccosine function. See Figure 10.

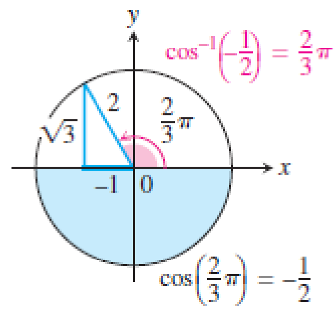



Figure 10

**Example 21**: If $\theta = \sin^{-1}\frac{\sqrt{3}}{2}$, find $\sin\theta$, $\sec\theta$, $\cot\theta$

**Solution**: $\sin\theta = \frac{\sqrt{3}}{2}$ $[\sin\theta = \sin(\sin^{-1}\frac{\sqrt{3}}{2})]$

$$\cos\theta = \frac{1}{2}$$

$$\sec\theta = \frac{1}{\frac{1}{2}} = 2$$

$$\cot\theta = \frac{1}{\sqrt{3}}$$

## Identities Involving Arcsine and Arccosine

(1) As we can see from Figure 11, the arccosine of $x$ satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi$$

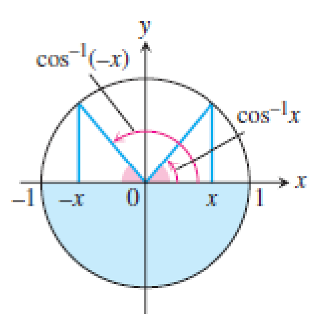


**Figure 11**

(2) Also, we can see from the triangle in Figure 12 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2$$

$$\tan^{-1} x + \cot^{-1} x = \pi/2$$

$$\sec^{-1} x + \csc^{-1} x = \pi/2$$

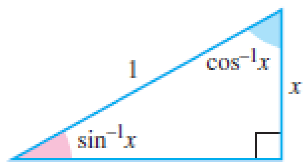



**Figure 12**

(3) $\sec^{-1} x = \cos^{-1} (1/x)$
$\csc^{-1} x = \sin^{-1} (1/x)$
$\cot^{-1} x = \tan^{-1} (1/x)$

## 7.13 Derivatives of Inverse Trigonometric Functions

### 1. The derivative of $y = \sin^{-1} x$

$y = \sin^{-1} x \rightarrow \sin y = \sin (\sin^{-1} x) \rightarrow \sin y = x$

$x = \sin y \rightarrow 1 = \cos y \, dy/dx \rightarrow dy/dx = 1 / \cos y$

$\sin^2 y + \cos^2 y = 1 \rightarrow \cos^2 y = 1 - \sin^2 y \rightarrow \cos y = \pm\sqrt{1 - sin^2 y}$

but $\sin^2 y = x^2 \rightarrow \cos y = \pm\sqrt{1 - x^2}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ or $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

If $y = \sin^{-1} u$, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$\boldsymbol{\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}}$$

## Derivatives of the inverse trigonometric functions

$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d(\cos^{-1} u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d(\cot^{-1} u)}{dx} = \frac{-1}{1+u^2}\frac{du}{dx}$$

$$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d(\csc^{-1} u)}{dx} = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \quad |u| > 1$$

**Example 22**: Find *dy/dx* for the function $y = x\sin^{-1}\sqrt{x} + \sqrt{x-1}$

**Solution**: $\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1-x}}\right)\cdot\frac{1}{2}x^{-\frac{1}{2}} + \sin^{-1}\sqrt{x}\,(1) + \frac{1}{2}(x-1)^{-1/2}$

$$= \frac{x}{2\sqrt{x}\sqrt{1-x}} + \sin^{-1}\sqrt{x} + \frac{1}{2\sqrt{x-1}}$$

**Example 23**: If $y = \tan^{-1}\frac{1-x}{1+x}$, find *dy/dx*

**Solution**: let $u = \frac{1-x}{1+x} \to y = \tan^{-1} u \to \frac{dy}{dx} = \frac{1}{1+u^2}\cdot\frac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{1+(\frac{1-x}{1+x})^2} \times \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} = \frac{-1-x-1+x}{(\frac{(1+x)^2+(1-x)^2}{(1+x)^2})(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-2(1+x)^2}{[(1+x)^2+(1-x^2)](1+x)^2} = \frac{-2}{[(1+x)^2+(1-x^2)]}$$

## 7.14 Integrals of the inverse trigonometric functions

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{-du}{\sqrt{1-u^2}} = \cos^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$

$$\int \frac{-du}{1+u^2} = \cot^{-1} u + C$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \int \frac{d(-u)}{(-u)\sqrt{u^2-1}} = \sec^{-1}|u| + C = \cos^{-1}\left|\frac{1}{u}\right| + C$$

$$\int \frac{-du}{u\sqrt{u^2-1}} = \int \frac{-d(-u)}{(-u)\sqrt{u^2-1}} = \csc^{-1}|u| + C = \sin^{-1}\left|\frac{1}{u}\right| + C$$

**Example 24**: Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$

Solution: $\sqrt{9-x^2} = \sqrt{9(1-\frac{x^2}{9})} = 3\sqrt{1-(\frac{x}{3})^2}$

Let $u = x/3 \rightarrow du = 1/3\, dx \rightarrow dx = 3\, du$

$$\int \frac{3du}{3\sqrt{1-u^2}} = \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}\left(\frac{x}{3}\right) + C$$

**Example 25**: Solve $x\sqrt{x^2-1}\frac{dy}{dx} = \sqrt{1-y^2}$, $y = -½$ when $x = 2$.

**Solution**: $x\sqrt{x^2-1}dy = \sqrt{1-y^2}dx \rightarrow \int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dy}{\sqrt{1-y^2}}$

$\sec^{-1}|x| + C = \sin^{-1} y \rightarrow \cos^{-1}\left|\frac{1}{x}\right| + C = \sin^{-1} y$

At $x = 2$, $y = -½ \rightarrow \cos^{-1}\left(\frac{1}{2}\right) + C = \sin^{-1}\left(-\frac{1}{2}\right)$

$\pi/3 + C = -\pi/6 \rightarrow C = -\pi/6 - \pi/3 = -\pi/2$

$$\sin^{-1} y = \cos^{-1}\left|\frac{1}{x}\right| - \frac{\pi}{2}$$

**Example 26**: Evaluate $\int_0^{\sqrt{2}/2} \frac{x\,dx}{\sqrt{1-x^4}}$

**Solution**: let $u = x^2 \rightarrow du = 2x\,dx \rightarrow x\,dx = du/2$

U.L. $= (\frac{\sqrt{2}}{2})^2 = \frac{2}{4} = \frac{1}{2}$, L.L. $= (0)^2 = 0$

$$\int_0^{1/2} \frac{du}{2\sqrt{1-u^2}} = \frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}[\sin^{-1} u]_0^{1/2}$$

$= ½ [\sin^{-1} ½ - \sin^{-1} 0] = ½ [\pi/6 - 0] = \pi/12$

**Example 27**: Evaluate $\int_0^{\ln\sqrt{3}} \frac{e^x dx}{1+e^{2x}}$

**Solution**:

Let $u = e^x \rightarrow du = e^x\,dx$

$1 + e^{2x} = 1 + (e^x)^2 = 1 + u^2$

U.L. $= e^{\ln\sqrt{3}} = \sqrt{3}$, L.L. $= e^0 = 1$

$$\int_1^{\sqrt{3}} \frac{du}{1+u^2} = [\tan^{-1} u]_1^{\sqrt{3}} = [\tan^{-1}\sqrt{3} - \tan^{-1} 1]$$

$= \pi/3 - \pi/4 = \pi/12$