CHAPTER 8

TECHNIQUES OF INTEGRATION

The Fundamental Theorem tells us how to evaluate a definite integral once we have an antiderivative for the integrand function. Table 8.1 summarizes the forms of antiderivatives for many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones.

In this chapter we study a number of other important techniques for finding antiderivatives (or indefinite integrals) for many combinations of functions whose antiderivatives cannot be found using the methods presented before.

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad$ (any number k)
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \dfrac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$
21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \quad (a > 0)$
22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \quad (x > a > 0)$

8.1 basic Integration Methods

1. Substitution Method

Example 1: Evaluate $\int \sqrt{1+x^2} \cdot x^5 dx$

Solution:

$$\int \sqrt{1+x^2} \cdot x^5 dx = \int \sqrt{1+x^2} \cdot x^4 \cdot x\, dx$$

Let $u = 1 + x^2 \rightarrow du = 2x\, dx \rightarrow dx = du/2$

$x^2 = u - 1 \rightarrow x^4 = (x^2)^2 = (u-1)^2$

$$\int \sqrt{u} \cdot (u-1)^2 \cdot \frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}} \cdot (u^2 - 2u + 1) du$$

$$= \frac{1}{2}\int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}) du = \frac{1}{2}[\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}] + C$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$

2. Completing the Square

To write the function ($ax^2 + bx + c$) in the form ($a u^2 + k$)

a. Factor out a from first two terms $\rightarrow a\left(x^2 + \frac{b}{a}x\right) + c$
b. Add and subtract the square of half coefficient of x

$$\left[\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \frac{b^2}{4a^2}\right]$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

c. Bring out the $\frac{-b^2}{4a^2}$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - a\left(\frac{b^2}{4a^2}\right) + c$$

$$a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+\left(c-\frac{b^2}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)=au^2+k$$

where: $u=x+\frac{b}{2a}, k=c-\frac{b^2}{4a}$

Example 2: Evaluate $\int\frac{dx}{x^2+2x+2}$

Solution:

$ax^2+2x+2=au^2+k;\ u=x+\frac{b}{2a}, k=\left(c-\frac{b^2}{4a}\right)$

$$a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$$

$a=1, b=2, c=2$

$$x+\frac{b}{2a}=\left(x+\frac{2}{2\times 1}\right)=(x+1)$$

$$\left(c-\frac{b^2}{4a}\right)=\left(2-\frac{2^2}{4\times 1}\right)=\left(2-\frac{4}{4}\right)=1$$

$$\int\frac{dx}{x^2+2x+2}=\int\frac{dx}{(x+1)^2+1}$$

Let $u=x+1\rightarrow du=dx\rightarrow\int\frac{du}{u^2+1}=\tan^{-1}u+c=\tan^{-1}(x+1)+c$

<u>3. Expanding a Power and using Trigonometric Identity</u>

Example 3: Evaluate $\int(\sec x+\tan x)^2dx$

Solution:

$$=\int(\sec^2x+2\sec x\tan x+\tan^2x)dx$$

$$=\int(\sec^2x+2\sec x\tan x+(\sec^2x-1))dx$$

$$=\int(2\sec^2x+2\sec x\tan x-1)dx=2\tan x+2\sec x-x+c$$

4. Reducing an improper fraction

Improper fraction is the fraction with degree of numerator equals or greater than the degree of denominator. The long division is used to integrate this fraction.

Example 4: Evaluate $\int \frac{dx}{\sqrt[3]{x}-\sqrt{x}}$

Solution: let $x = u^6 \rightarrow dx = 6\,u^5 \rightarrow \sqrt[3]{x} = (u^6)^{1/3} = u^2$

$$\sqrt{x} = (u^6)^{1/2} = u^3$$

$$\int \frac{6u^5 du}{u^2 - u^3} = 6\int \frac{u^5 du}{u^2(1-u)} = 6\int \frac{u^3}{1-u}\ [improper\ fraction]$$

$$= 6\int \frac{-u^3 du}{-(1-u)} = -6\int \frac{u^3 du}{u-1}$$

$$= -6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du$$

$$= -6\left[\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right] + c$$

$$= -6\left[\frac{1}{3}\sqrt{x} + \frac{1}{2}\sqrt[3]{x} + \sqrt[6]{x} + \ln\left|\sqrt[6]{x} - 1\right|\right] + c$$

5. Separating a Fraction

Example 5: Evaluate $\int \frac{x^3+3}{x} dx$

Solution:

$$\int \frac{x^3+3}{x} dx = \int \left(x + \frac{3}{x}\right) dx$$

$$= \frac{1}{2}x^2 + 3\ln|x| + c$$

6. Sequences of Substitutions

Example 6: Evaluate $\int \sqrt{1 + sin^2(x-1)} \cdot \sin(x-1).\cos(x-1)dx$

Solution:

Let $u = \sin(x-1) \rightarrow du = \cos(x-1)\, dx$

$$\int \sqrt{1+u^2} \cdot u\ \cdot du$$

Let $v = 1 + u^2 \rightarrow dv = 2\,u\,du \rightarrow u\,du = dv/2$

$$\int \sqrt{v}\,\frac{dv}{2} = \frac{1}{2}\int v^{1/2}\,dv = \frac{1}{2}\left(\frac{2}{3}\right)v^{3/2} + c$$

$$= \frac{1}{3}v^{3/2} + c = \frac{1}{3}(1+u^2)^{3/2} + c = \frac{1}{3}[1 + sin^2(x-1)]^{3/2} + c$$

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)dx$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty.

If $u = f(x)$ and $v = g(x) \rightarrow \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

In general $d\,(uv) = u\,dv + v\,du$

$u\,dv = d\,(uv) - v\,du \rightarrow \int u\,dv = \int d\,(uv) - \int v\,du$

$$\int \boldsymbol{u\,dv = u\,v -} \int \boldsymbol{v\,du}$$

Formula for integrating by parts

Note:

1. (u) is chosen in which can b differentiated repeatedly to become zero, or chosen in which can be appear repeatedly after differentiation.
2. (dv) is chosen in which can be integrated repeatedly without difficulty.

Example 7: Find

$$\int x\cos x\;dx$$

Solution:

We use the formula $\int u\,dv = u\,v - \int v\,du$

$$u = x, \qquad dv = cos\,x\,dx,$$
$$du = dx, \qquad v = sin\,x,$$

Then

$$\int x\cos x\;dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

Example 8: Find

$$\int \ln x\,dx$$

Solution:
Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = u\,v - \int v\,du$

$u = \ln x$ (*simplifies when differentiated*) $dv = dx$ (*easy to integrate*)
$du = (1/x)\,dx$ $v = x$ (*simplest integration*)

Then $\int u\,dv = u\,v - \int v\,du$ will be:

$$\int \ln x\,dx = x\ln x - \int x\,\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$$

Example 9: Evaluate

$$\int x^2 e^x\,dx$$

Solution: $\int u\,dv = u\,v - \int v\,du$
$u = x^2$, $dv = e^x\,dx$,
$du = 2x\,dx$ and $v = e^x$
then :

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x\,e^x\,dx$$

We need to repeat the integration by parts for the right term ($\int x\,e^x\,dx$) with:
$u = x$, $dv = e^x\,dx$,
$du = dx$ and $v = e^x$
then:

$$\int x\,e^x\,dx = x\,e^x - \int e^x dx = xe^x - e^x + C$$

Using this last evaluation, we then obtain:

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x\,e^x\,dx$$
$$= x^2 e^x - 2xe^x + 2e^x + C$$

Example 10: Evaluate

$$\int e^x \cos x\,dx$$

Solution:

Let $u = e^x$ and $dv = \cos x\,dx$

Then $du = e^x\,dx$, $v = \sin x$,

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

The second integral is like the first except that it has sin *x* in place of cos *x*. To evaluate it, we use integration by parts with

$u = e^x$, $dv = \sin x\,dx$, $du = e^x\,dx$, $v = -\cos x$,

$$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x - \int \quad (-\cos x)(e^x\,dx)\right)$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give:

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C1$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C$$

8.3 Tabular Integration

This integration is used when the integration by parts required many repititions

Example 11: Evaluate

$$\int x^2 e^x\, dx$$

Solution:

With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives	$g(x)$ and its integrals
x^2	e^x
$2x$	e^x
2	e^x
0	e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x\, dx = x^2 e^x - 2x e^x + 2 e^x + c$$

Example 12: Evaluate

$$\int x^3 \sin x\, dx$$

Solution:

With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives	$g(x)$ and its integrals
x^3	$\sin x$
$3x^2$	$-\cos x$
$6x$	$-\sin x$
6	$\cos x$
0	$\sin x$

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$$

8.4 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\, dx = \tan x + c$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

8.4.1 Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero)

Case 1: <u>*m* is odd</u>

Save one sine factor and use $\sin^2 x = 1 - \cos^2 x$. Then substitute $u = \cos x$

Example 13: Evaluate

$$\int \sin^5 x \cos^2 x\, dx$$

Solution:

$\int \sin^5 x \,.\, \cos^2 x\, dx = \int \sin x \,.\, \sin^4 x \,.\, \cos^2 x\, dx$

$= \int \sin x \,.\, (1 - \cos^2 x)^2 \,.\, \cos^2 x\, dx$

Let $u = \cos x \rightarrow du = -\sin x\, dx \rightarrow \sin x\, dx = -du$

$= -\int (1 - u^2)^2 .\, u^2 du = -\int (1 - 2u^2 + u^4).\, u^2\, du$

$= -\int (u^2 - 2u^4 + u^6)\, du = -[(1/3)\, u^3 - (2/5)\, u^5 + (1/7)\, u^7] + c$

$= -(1/3) \cos^3 x - (2/5) \cos^5 x - (1/7) \cos^7 x + c$

Case 2: <u>n is odd</u>

Save one cosine factor and use ($\cos^2 x = 1 - \sin^2 x$). Then substitute $u = \sin x$

Example 14: Evaluate

$$\int \cos^5 x \, dx$$

Solution:

$$\int \cos^5 x \, dx = \int \cos x \, . \cos^4 x \, dx = \int \cos x \, (1 - \sin^2 x)^2 \, dx$$

Let $u = \sin x \to du = \cos x \, dx$

$$\int (1 - u^2)^2 . \, du = -\int (1 - 2u^2 + u^4) \, du = u - (2/3) \, u^3 + (1/5) \, u^5 + c$$
$$= \sin x - (2/3) \sin^3 x + (1/5) \sin^5 x + c$$

Case 3: <u>m and n are even</u>

Use: $sin^2 x = \frac{1-\cos 2x}{2}$

$cos^2 x = \frac{1+\cos 2x}{2}$

Example 15: Evaluate $\int \sin^2 x \cos^4 x \, dx$

Solution: $\int \sin^2 x \cos^4 x \, dx = \left(\frac{1-\cos 2x}{2}\right) \cdot \left(\frac{1+\cos 2x}{2}\right)^2 \cdot dx$

$= 1/8 \int (1 - \cos 2x) \, . \, (1 + 2 \cos 2x + \cos^2 2x). \, dx$

$= 1/8 \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x). \, dx$

Now: $\int \cos^2 2x \, dx = (1/2) \int (1 + \cos 4x) \, dx = (½) \, [x + (1/4) \sin 4x]$

$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx$

Let $u = sin\ 2x \to du = 2\ cos\ 2x\ dx \to cos\ 2x\ dx = du/2$

$$= (1/2) \int (1 - u^2) \, du = ½ \, [u - (1/3) \, u^3] = ½ \, [\sin 2x - (1/3) \sin^3 2x]$$

$\int \sin^2 x \cos^4 x \, dx = (1/8) \, [\, x + (1/2) \sin 2x - (1/2) \, (x + (1/4) \sin 4x) - (1/2) \, (\sin 2x - (1/3) \sin^3 2x] + c$

$= (1/8)\ [\ x + (1/2) \sin 2x - (1/2)\ x - (1/8) \sin 4x - (1/2) \sin 2x + (1/6) \sin^3 2x] + c$

$= (1/16)\ [\ x - (1/4) \sin 4x + (1/3) \sin^3 2x + c$

Note: if both m and n are odd, use either Case 1 or Case 2.

8.4.2 Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\, dx, \quad \int \sin mx \cos nx\, dx, \text{ and } \quad \int \cos mx \sin nx\, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \tfrac{1}{2} [\cos (m - n)\, x - \cos (m + n)\, x],$$

$$\sin mx \cos nx = \tfrac{1}{2} [\sin (m - n)\, x + \sin (m + n)\, x],$$

$$\cos mx \cos nx = \tfrac{1}{2} [\cos (m - n)\, x + \cos (m + n)\, x],$$

Example 16: Evaluate $\int \sin 2x \sin x\, dx$

Solution: $m = 2$, $n = 1$

$\int \sin 2x \sin x\, dx = \int \tfrac{1}{2} [\cos x - \cos 3x]\, dx$

$$= \tfrac{1}{2} \int \cos x\, dx - \tfrac{1}{2} \int \cos 3x\, dx = \tfrac{1}{2} \sin x - 1/6 \sin 3x + c$$

Example 17: Evaluate $\int \sin 3x \cos 5x\, dx$
Solution:
$m = 3$, $n = 5$
$\int \sin 3x \cos 5x\, dx = \tfrac{1}{2} \int [\sin (3 - 5)\, x + \sin (3 + 5)\, x]\, dx$
$= \tfrac{1}{2} \int [\sin (-2x) + \sin (8x)]\, dx$

$= \frac{1}{2} \int [\sin 8x - \sin 2x]\, dx \qquad (\sin(-x) = -\sin x)$
$= \frac{1}{2} [(-1/8) \cos 8x + (1/2) \cos 2x] + C$
$= \frac{1}{4} \cos 2x - (1/16) \cos 8x + C$

8.4.3 Integrals of Powers of tan *x* and sec *x*

Case 1: Odd Power of Secant
Use integration by parts and the identity ($\tan^2 x = \sec^2 x - 1$)

Example 18: Evaluate

$$\int \sec^3 x\, dx$$

Solution:
$\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx$
Let $u = \sec x \rightarrow du = \sec x \tan x\, dx$
$dv = \sec^2 x\, dx \rightarrow v = \tan x$

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int \tan x .\sec x .\tan x\, dx \\ &= \sec x \tan x - \int \tan^2 x .\sec x\, dx \\ &= \sec x \tan x - \int (\sec^2 x - 1) .\sec x\, dx \\ &= \sec x \tan x - \int (\sec^3 x - \sec x)\, dx \\ &= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx\end{aligned}$$

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln |\sec x \tan x|$$

$$\int \sec^3 x\, dx = \tfrac{1}{2} [\sec x \tan x + \ln |\sec x \tan x|] + C$$

Case 2: Even Power of Secant

Save $\sec^2 x$ and use ($\sec^2 x = \tan^2 x + 1$)

Example 19: Evaluate

$$\int \sec^4 x\, dx$$

Solution:

$\int \sec^4 x\, dx = \int \sec^2 x. \sec^2 x\, dx = \int (\tan^2 x + 1) . \sec^2 x\, dx$

$= \int [\tan^2 x . \sec^2 x + \sec^2 x]\, dx = (1/3) \tan^3 x + \tan x + C$

Case 3: Odd and Even Power of Tangent

Save $\tan^2 x$ and use ($\tan^2 x = \sec^2 x - 1$)

Example 20: Evaluate

$$\int \tan^5 3x\, dx$$

Solution:

$\int \tan^5 3x\, dx = \int \tan^2 3x. \tan^3 3x\, dx = \int (\sec^2 3x - 1) \tan^3 3x\, dx$
$= \int \tan^3 3x \sec^2 3x\, dx - \int \tan^3 3x\, dx$
$= (1/3) \int (\tan 3x)^3 . 3 \sec^2 3x\, dx - \int \tan^3 3x\, dx$
$= (1/3) \int (\tan 3x)^3 . 3 \sec^2 3x\, dx - \int \tan^3 3x\, dx$
$= (1/3) (\tan^4 3x)/4 - \int (\sec^2 3x - 1) \tan 3x\, dx$
$= (1/12) \tan^4 3x - \int \tan 3x \ \sec^2 3x\, dx - \int \tan 3x\, dx$
$= (1/12) \tan^4 3x - (1/3) \int \tan 3x. 3 \sec^2 3x\, dx - \int \tan 3x\, dx$
$= (1/12) \tan^4 3x - (1/3) \ (\tan^2 3x)/2 - (1/3) \ln |\sec 3x| + C$
$= (1/12) \tan^4 3x - (1/6) \ (\tan^2 3x) - (1/3) \ln |\sec 3x| + C$

8.4.4 Power Products of tangent and secant

$\int \sec^m x . \tan^n x . dx$, m, n are positive

Case 1: m is even
Save $\sec^2 x$ and use ($\sec^2 x = \tan^2 x + 1$), then substitute $u = \tan x$.

Example 21: Evaluate

$$\int \sec^4 x . \tan x\, dx$$

Solution:

$\int \sec^4 x . \tan x\, dx = \int \sec^2 x . \sec^2 x . \tan x\, dx$
$= \int (\tan^2 x + 1) . \sec^2 x . \tan x\, dx$
Let $u = \tan x \rightarrow du = \sec^2 x\, dx$
$\int (u^2 + 1) . u\, du = \int (u^3 + u)\, du = (1/4)\, u^4 + (1/2)\, u^2 + C$
$= (1/4) \tan^4 x + (1/2) \tan^2 x + C$

Case 2: *m* and *n* is odd
Save sec x tan x and use ($\tan^2 x = \sec^2 x - 1$) for remaining factor, then substitute $u = \sec x$.

Example 22: Evaluate

$$\int \sec^3 x \,.\, \tan^3 x \, dx$$

Solution:

$\int \sec^3 x \,.\, \tan^3 x \, dx = \int (\sec x \tan x) \,.\, \sec^2 x \,.\, \tan^2 x \, dx$
$= \int (\sec x \tan x) \,.\, \sec^2 x \,.\, (\sec^2 x - 1) \, dx$
Let $u = \sec x \rightarrow du = \sec x \tan x \, dx$
$\int u^2 \,.\, (u^2 - 1) \, du = \int (u^4 - u^2) \, du = (1/5)\, u^5 - (1/3)\, u^3 + C$
$= (1/5) \sec^5 x - (1/3) \sec^3 x + C$

Case 3: *m* is odd and *n* is even

Use ($\tan^2 x = \sec^2 x - 1$)

Example 23: Evaluate

$$\int \sec x \,.\, \tan^2 x \, dx$$

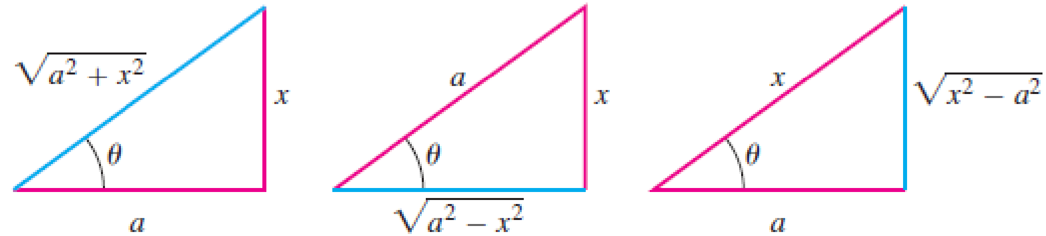
$\int \sec x \,.\, \tan^2 x \, dx = \int \sec x \,.\, (\sec^2 x - 1) \, dx = \int (\sec^3 x - \sec x) \, dx$

$\int \sec^3 x \, dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x \tan x|]$ [from last example]

$\int \sec x \,.\, \tan^2 x \, dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x \tan x|] - \ln |\sec x \tan x| + C$

8.5 Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$ and $x = a \sec \theta$. These substitutions are effective in transforming integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$ and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly since they come from the reference right triangles in Figure 1.



$x = a \tan \theta$ $\quad$ $x = a \sin \theta$ $\quad$ $x = a \sec \theta$

$\sqrt{a^2 + x^2} = a|\sec \theta|$ $\quad$ $\sqrt{a^2 - x^2} = a|\cos \theta|$ $\quad$ $\sqrt{x^2 - a^2} = a|\tan \theta|$

With $x = a \tan \theta$,

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2 (1 + \tan^2 \theta) = a^2 \sec^2 \theta$$

With $x = a \sin \theta$,

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2 (1 - \sin^2 \theta) = a^2 \cos^2 \theta$$

With $x = a \sec \theta$,

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2 (\sec^2 \theta - 1) = a^2 \tan^2 \theta$$

For $a^2 - u^2$ use $u = a \sin \theta$, $\quad -\pi/2 \leq \theta \leq \pi/2$

For $a^2 + u^2$ use $u = a \tan \theta$, $\quad -\pi/2 \leq \theta \leq \pi/2$

For $u^2 - a^2$ use $u = a \sec \theta$, $\quad 0 \leq \theta \leq \pi,\ \theta \neq \pi/2$

Example 24: Evaluate

$$\int \frac{dx}{\sqrt{1 - 4x^2}}$$

Solution:

$1 - 4x^2 = a^2 - u^2$, $a = 1$, $u = 2x$

Use $u = a \sin \theta \rightarrow 2x = 1 \sin \theta$,

$\theta = \sin^{-1} 2x \quad -\pi/2 \leq \theta \leq \pi/2$

$2\, dx = \cos \theta\, d\theta \rightarrow dx = ½ \cos \theta\, d\theta$

$$\int \frac{dx}{\sqrt{1 - 4x^2}} = \int \frac{\frac{1}{2}\cos\theta\, d\theta}{\sqrt{1 - sin^2\theta}} = \frac{1}{2}\int \frac{\cos\theta\, d\theta}{\sqrt{cos^2\theta}} = \frac{1}{2}\int \frac{\cos\theta\, d\theta}{\pm\cos\theta}$$

But cos θ is + *ve* for $-\pi/2 \le \theta \le \pi/2$

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{\cos\theta\, d\theta}{\cos\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C = \frac{1}{2}\sin^{-1} 2x + C$$

Example 24: Evaluate

$$\int \frac{dy}{25+16y^2}$$

Solution:

$25 + 16\,y^2 = a^2 + u^2, \quad a = 5, u = 4y$
Use $u = a \tan\theta \rightarrow 4y = 5 \tan\theta$
$\theta = \tan^{-1} 4y/5 \quad -\pi/2 \le \theta \le \pi/2$
$4\,y = 5 \tan\theta \rightarrow 4\,dy = 5 \sec^2\theta\, d\theta \rightarrow dy = (5/4)\sec^2\theta\, d\theta$

$$\int \frac{dy}{25+16y^2} = \int \frac{\frac{5}{4}sec^2\theta\, d\theta}{25+25\,tan^2\theta} = \frac{5}{4}\int \frac{sec^2\theta\, d\theta}{25\,sec^2\theta}$$

$= (1/20)\int d\theta = (1/20)\,\theta + C = (1/20)\tan^{-1}(4y/5) + C$

Example 25: Evaluate

$$\int \frac{(2x+3)dx}{4x^2+4x+5}$$

Solution:
$4\,x^2 + 4x + 5 = a\,x^2 + bx + c = a\,u^2 + k \;, a = 4, b = 4, c = 5$
$u = x + b/2a = x + u/\,2*u$
$u = x + ½$
$k = c - (b^2/4a) = 5 - (4^2/4*4) = 5 - 1 = 4$
$4\,x^2 + 4x + 5 = 4\,(x + ½)^2 + 4 = 4\,[(x + ½)^2 + 1]$

$$\int \frac{(2x+3)dx}{4x^2+4x+5} = \int \frac{(2x+3)dx}{4[\left(x+\frac{1}{2}\right)^2+1]} = \frac{1}{4}\int \frac{(2x+3)dx}{\left(x+\frac{1}{2}\right)^2+1}$$

$(x + ½)^2 + 1 = u^2 + a^2 \;, u = x + ½ \;, a = 1$

Use $u = a \tan\theta \rightarrow x + ½ = 1 \tan\theta \rightarrow dx = \sec^2\theta\, d\theta$

$x + ½ = \tan\theta \rightarrow \theta = \tan^{-1}(2x+1)/2 \quad -\pi/2 \le \theta \le \pi/2$

$2x = 2\tan\theta - 1$

$$\frac{1}{4}\int \frac{(2x+3)dx}{\left(x+\frac{1}{2}\right)^2+1} = \frac{1}{4}\int \frac{(2\tan\theta - 1 + 3)\, sec^2\theta\, d\theta}{tan^2\theta + 1}$$

$$\frac{1}{4}\int \frac{(2\tan\theta + 2)\, sec^2\theta\, d\theta}{sec^2\theta} = \frac{1}{4}\int 2(\tan\theta + 1)\, d\theta = \frac{1}{2}\int \tan\theta\, d\theta + \frac{1}{2}\int d\theta$$

$= ½ \ln|\sec\theta| + ½\,\theta + C$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+4x+5}}{2}\right| + \frac{1}{2}\tan^{-1}\left(\frac{2x+1}{2}\right) + C$$

8.6 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x-3)/(x^2-2x-3)$ can be rewritten as:

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

The method for rewriting rational functions as a sum of simpler fractions is called the **method of partial fractions**. In the case of the preceding example, it consists of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

Case 1: Distinct Linear Factors of $g(x)$:

$$\frac{f(x)}{g(x)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3} + \cdots$$

Example 26: Evaluate

$$\int \frac{(6x+1)dx}{x^2+4x-5}$$

Solution:

$$\frac{(6x+1)}{x^2+4x-5} = \frac{(6x+1)}{(x-1)(x+5)} = \frac{A}{(x-1)} + \frac{B}{(x+5)} = \frac{A(x+5)+B(x-1)}{(x-1)(x+5)}$$

Multiplying both sides with $(x – 1)\ (x + 5)$

$6x + 1 = A\ (x + 5) + B\ (x – 1) \rightarrow 6x + 1 = A\ x + 5\ A + B\ x – B$

$6x + 1 = (A + B)\ x + (5A – B)$

$A + B = 6,\ 5A – B = 1$

$A = 6 – B \rightarrow 5\ (6 – B) – B = 1 \rightarrow 30 – 5B – B = 1 \rightarrow 6\ B = 29 \rightarrow B = 29/6$

$A = 6 – (29/6) = 7/6$

$$\int \frac{(6x+1)dx}{x^2+4x-5} = \int \frac{\frac{7}{6}}{x-1}dx + \frac{\frac{29}{6}}{x+5}dx = \frac{7}{6}\int \frac{dx}{x-1} + \frac{29}{6}\int \frac{dx}{x+5}$$

$= (7/6)\ \ln\ |x – 1| + (29/6)\ \ln\ |x – 1| + C$

Example 27: Evaluate

$$\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\ dx$$

Solution:

Use long division:

$$\frac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \frac{x+5}{x^2-2x-8}$$

$$\frac{x+5}{x^2-2x-8}=\frac{x+5}{(x+2)(x-4)}$$

$$\frac{x+5}{(x+2)(x-4)}=\frac{A}{x+2}+\frac{B}{x-4}$$

Multiplying both sides by $(x + 2)\ (x - 4)$
$x + 5 = A\ (x - 4) + B\ (x + 2)$
$x + 5 = Ax - 4A + Bx + 2B$
$x + 5 = (A+ B)x - 4A + 2B$
$A + B = 1 \quad (\times -2) \rightarrow - 2A - 2B = - 2$

$- 4A + 2B = 5$
$- 2A - 2B = -2$
By adding:
$- 6\ A = 3$

$A = - ½$, $B = 1 - (- ½) = 3/2$

$$\int\frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx=\int\left[2x+\frac{-\frac{1}{2}}{x+2}+\frac{\frac{3}{2}}{x-4}\right]dx$$

$= x^2 - ½ \ln \left| x + 2 \right| + (3/2) \ln \left| x + 2 \right| + k$

Case 2: Repeated Linear Factors of $g\ (x)$:

$$\frac{f(x)}{g(x)}=\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}+\cdots$$

Example 28: Evaluate

$$\int \frac{x\,dx}{x^2 + 2x + 1}$$

Solution:

$$\frac{x}{x^2 + 2x + 1} = \frac{x}{(x+1)(x+1)} = \frac{x}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$$

$x = A\,(x + 1) + B \rightarrow x = Ax + (A + B)$

$A = 1, A + B = 0 \rightarrow A = -B \rightarrow B = -A = -1$

$$\int \frac{x\,dx}{x^2 + 2x + 1} = \int \frac{1\,dx}{x+1} - \int \frac{1\,dx}{(x+1)^2} = \ln|x+1| - \frac{(x+1)^{-1}}{-1} + C$$

$$= \ln|x+1| - \frac{1}{x+1} + C$$

Example 29: Evaluate

$$\int \frac{5x^2 + 20\,x + 6}{x^3 + 2\,x^2 + x}\,dx$$

Solution:

$$\frac{5x^2 + 20\,x + 6}{x^3 + 2\,x^2 + x} = \frac{5x^2 + 20\,x + 6}{x\,(x^2 + 2x + 1)} = \frac{5x^2 + 20\,x + 6}{x\,(x+1)^2}$$

$\frac{5x^2+20\,x+6}{x\,(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ (both case 1 and case 2)

Multiply both sides with $x\,(x + 1)^2$:

$5\,x^2 + 20\,x + 6 = A\,(x + 1)^2 + Bx\,(x + 1) + Cx$

$5\,x^2 + 20\,x + 6 = A\,(x^2 + 2\,x + 1) + Bx^2 + Bx + Cx$

$5x^2 + 20x + 6 = Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$

$5x^2 + 20x + 6 = (A + B)x^2 + (2A + B + C)x + A$

$A = 6,\ A + B = 5 \rightarrow B = -1,\ 2A + B + C = 20 \rightarrow C = 9$

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\, dx = \int \left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right] dx$$

$$6\ln|x| - \ln|x+1| + \frac{9}{x+1} + k$$

Case 3: Distinct Irreducible Quadratic Factors of $g(x)$:

$$\frac{f(x)}{g(x)} = \frac{Ax + B}{a_1x^2 + b_1x + c_1} + \frac{Cx + D}{a_2x^2 + b_2x + c_2} + \frac{Ex + F}{a_3x^2 + b_3x + c_3} + \cdots$$

Example 30: Evaluate

$$\int \frac{x-3}{(x^2+1)(x-1)^2}\, dx$$

Solution:

$$\frac{x-3}{(x^2+1)(x-1)^2} = \frac{Ax + B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$

Multiply both sides with $(x^2 + 1)(x - 1)^2$

$x - 3 = Ax + B(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$

$x - 3 = Ax + B(x^2 - 2x + 1) + Cx - C(x^2+1) + D(x^2+1)$

$x - 3 = Ax^3 - 2Ax^2 + Ax + Bx^2 - 2Bx + B + Cx^3 + Cx - C + Dx^2 + D$

$x - 3 = (A+C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + (B - C + D)$

$A + C = 0$ ………(1)

–2A + B – C + D = 0 ……..(2)

A – 2B + C = 1 ……….(3)

B – C + D = – 3 ……….(4)

Equ. *2 & 4 : → – 2A + (– 3) = 0 → – 2A = 0 – (– 3) → – 2A = 3 → A = – 3/2*

From Equ *1: C = – A = 3/2*

From Equ. *3: A – 2B + C = 1 → –(3/2) – 2B +(3/2) = 1 → 2B = – 1 → B = – ½*

From Equ. 3 : *– ½ - (3/2) + D = – 3 → (– 4/2) + D = – 3 → D = – 3 + 2 = –1*

$$\int \frac{(x-3)dx}{(x^2+1)(x-1)^2} = \frac{\left(\frac{-3}{2}x - \frac{1}{2}\right)\,dx}{(x^2+1)} + \int \frac{\frac{3}{2}dx}{(x-1)} - \int \frac{dx}{(x-1)^2}$$

$$= -\frac{3}{2}\int \frac{x\,dx}{(x^2+1)} - \frac{1}{2}\int \frac{dx}{x^2+1} + \frac{3}{2}\int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2}$$

$$= -\frac{3}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + \frac{3}{2}\ln|x-1| + \frac{1}{(x-1)} + C$$

Case 4: Repeated Irreducible Quadratic Factors of *g* (*x*)*:*

$$\frac{f(x)}{g(x)} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{(ax^2+bx+c)^3} + \cdots$$

Example 31: Evaluate

$$\int \frac{1-x+2x^2-x^3}{x^5+2x^3+x}\,dx$$

Solution:

$$\frac{1-x+2x^2-x^3}{x^5+2x^3+x} = \frac{1-x+2x^2-x^3}{x(x^4+2x^2+1)} = \frac{1-x+2x^2-x^3}{x(x^2+1)^2}$$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiply both sides with $x\,(x^2+1)^2$

$1 - x + 2x^2 - x^3 = A\,(x^2+1)^2 + (B\,x + C)\,(x)\,(x^2+1) + (D\,x + E)\,(x)$

$1 - x + 2x^2 - x^3 = A\,(x^4 + 2x^2 + 1) + (B\,x^2 + C\,x)\,(x^2+1) + Dx^2 + Ex$

$1 - x + 2x^2 - x^3 = Ax^4 + 2Ax^2 + A + B\,x^4 + B\,x^2 + C\,x^3 + C\,x + Dx^2 + Ex$

$1 - x + 2x^2 - x^3 = (A+B)\,x^4 + C\,x^3 + (2A + B + D)\,x^2 + (C + E)\,x + A$

$A = 1\,,\ A + B = 0 \rightarrow B = -1, \quad C = -1$

$2A + B + D = 2 \rightarrow D = 1$

$C + E = -1 \rightarrow E = 0$

$$\int \frac{1-x+2x^2-x^3}{x^5+2x^3+x}\,dx = \int \left[\frac{1}{x} + \frac{(-x-1)}{x^2+1} + \frac{x}{(x^2+1)^2}\right] dx$$

$$= \int \left[\frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\right] dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1} x - \frac{1}{2(x^2+1)} + k$$

8.7 The Substitution Z = tan (*x*/2)

The substitution z = tan (x/2) reduces the problem of integrating any rational function of sin x and cos x to a problem involving a rational function of z

$z = \tan(x/2)$, $\cos^2\theta = \frac{1}{2}(1 + \cos\theta)$, $\cos^2(x/2) = \frac{1}{2}(1 + \cos x)$

$2\cos^2(x/2) = 1 + \cos x \rightarrow \cos x = 2\cos^2(x/2) - 1$

$$\cos x = \frac{2}{sec^2\frac{x}{2}} - 1 = \frac{2}{tan^2\frac{x}{2}} - 1 = \frac{2}{z^2+1} = \frac{2 - z^2 - 1}{z^2+1}$$

$$\mathbf{\cos x = \frac{1 - z^2}{z^2+1}}$$

$\sin 2\theta = 2\sin\theta\cos\theta$, $\sin x = 2\sin(x/2)\cos(x/2)$

$$\sin x = \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}$$

$$cos^2\frac{x}{2} = 2\tan\frac{x}{2}.\frac{1}{sec^2\frac{x}{2}}$$

$$= 2\tan\frac{x}{2}.\frac{1}{tan^2\frac{x}{2}+1} = 2\,z\;\frac{1}{z^2+1}$$

$$= \mathbf{\frac{2\,z}{z^2+1}}$$

$z = \tan(x/2) \rightarrow dz = \frac{1}{2}\sec^2(x/2)\,dx \rightarrow dx = \frac{2\,dz}{sec^2\frac{x}{2}} \rightarrow dx = \frac{2\,dz}{tan^2\frac{x}{2}+1}$

$$\boldsymbol{dx = \frac{2\,dz}{z^2+1}}$$

Example 32: Evaluate

$$\int \frac{dx}{1-\sin x}$$

Solution:

$$\int \frac{dx}{1-\sin x} = \int \frac{\frac{z\,dz}{z^2+1}}{1-\frac{2z}{z^2+1}} = \int \frac{\frac{z\,dz}{z^2+1}}{\frac{z^2+1-2z}{z^2+1}} = 2\int \frac{dz}{z^2-2z+1}$$

$$= 2\int \frac{dz}{(z-1)^2} = -\frac{2}{z-1} + C = -\frac{2}{\tan\left(\frac{x}{2}\right)-1} + C$$