Notes For the Level 1 Lecture Course in Fluid Mechanics

Department of Civil Engineering, University of Tikrit.
FLUID MECHANICS
Ass. Proff. Yaseen Ali Salih
2016

1. **Contents of the module**
2. ***Objectives:***
   - The course will introduce fluid mechanics and establish its relevance in civil engineering.
   - Develop the fundamental principles underlying the subject.
   - Demonstrate how these are used for the design of simple hydraulic components.
3. ***Consists of:***
   - Lectures:
     30 Classes presenting the concepts, theory and application.
     Worked examples will also be given to demonstrate how the theory is applied.
   -
   - Laboratories: 1 x 3 hours
     These laboratory sessions examine how well the theoretical analysis of fluid dynamics describes what we observe in practice.
     During the laboratory you will take measurements and draw various graphs according to the details on the laboratory sheets. These graphs can be compared with those obtained from theoretical analysis.
     You will be expected to draw conclusions as to the validity of the theory based on the results you have obtained and the experimental procedure.
     After you have completed the two laboratories you should have obtained a greater understanding as to how the theory relates to practice, what parameters are important in analysis of fluid and where theoretical predictions and experimental measurements may differ.
     The two laboratories sessions are:
     1. Impact of jets on various shaped surfaces - a jet of water is fired at a target and is deflected in various directions. This is an example of the application of the momentum equation.
     2. The rectangular weir - the weir is used as a flow measuring device. Its accuracy is investigated. This is an example of how the Bernoulli (energy) equation is applied to analyses fluid flow.

[As you know, these laboratory sessions are **compulsory** course-work. You must attend them. Should you fail to attend either one you will be asked to complete some extra work. This will involved a detailed report and further questions. The simplest strategy is to do the lab.]

- Homework:
  Example sheets: These will be given for each section of the course. Doing these will **greatly** improve your exam mark. They are course work but do not have credits toward the module.
  Lecture notes: Theses should be studied but explain only the basic outline of the necessary concepts and ideas.
  Books: It is very important do some extra reading in this subject. To do the examples you will definitely need a text book. Any one of those identified below is adequate and will also be useful for the fluids courses in higher years.
    - Example classes:
      There will be example classes each week. You may bring any problems/questions you have about the course and example sheets to these classes.
-

4. ***Specific Elements:***
   - Introduction
   - Fluid Properties
     - Fluids vs. Solids
     - Viscosity
     - Newtonian Fluids
     - Properties of Fluids
   - Statics
     - Hydrostatic pressure
     - Manometry / pressure measurement
     - Hydrostatic forces on submerged surfaces
   - Dynamics
     - The continuity equation.
     - The Bernoulli Equation.
     - Applications of the Bernoulli equation.
     - The momentum equation.
     - Application of the momentum equation.
   - Real Fluids
     - Boundary layer.
     - Laminar flow in pipes.
   - Introduction to dimensional analysis
     - Dimensional analysis
     - Similarity
5. ***Books:***

   Any of the book listed below are more than adequate for this module.

   (You will probably not need any more fluid mechanics books on the rest of the Civil Engineering course)

Mechanics of Fluids, Massey B S., Van Nostrand Reinhold.

Fluid Mechanics, Douglas J F, Gasiorek J M, and Swaffield J A, Longman.

Civil Engineering Hydraulics, Featherstone R E and Nalluri C, Blackwell Science.

Hydraulics in Civil and Environmental Engineering, Chadwick A, and Morfett J., E & FN Spon - Chapman & Hall.

6- *Civil Engineering Fluid Mechanics*

Why are we studying fluid mechanics on a Civil Engineering course? The provision of adequate water services such as the supply of potable water, drainage, sewerage are essential for the development of industrial society. It is these services which civil engineers provide.

Fluid mechanics is involved in nearly all areas of Civil Engineering either directly or indirectly. Some examples of direct involvement are those where we are concerned with manipulating the fluid:

- Sea and river (flood) defences;
- Water distribution / sewerage (sanitation) networks;
- Hydraulic design of water/sewage treatment works;
- Dams;
- Irrigation;
- Pumps and Turbines;
- Water retaining structures.

And some examples where the primary object is construction - yet analysis of the fluid mechanics is essential:

- Flow of air in / around buildings;
- Bridge piers in rivers;
- Ground-water flow.

Notice how nearly all of these involve water. The following course, although introducing general fluid flow ideas and principles, will demonstrate many of these principles through examples where the fluid is water.

7-System of units

As any quantity can be expressed in whatever way you like it is sometimes easy to become confused as to what exactly or how much is being referred to. This is particularly true in the field of fluid mechanics. Over the years many different ways have been used to express the various quantities involved. Even today different countries use different terminology as well as different units for the same thing - they even use the same name for different things e.g. an American pint is 4/5 of a British pint!

To avoid any confusion on this course we will always used the SI (metric) system - which you will already be familiar with. It is essential that all quantities be expressed in the same system or the wrong solution will results.

Despite this warning you will still find that that this is the most common mistake when you attempt example questions.

6. *The SI System of units*

The SI system consists of six **primary** units, from which all quantities may be described. For convenience **secondary** units are used in general practise which are made from combinations of these primary units.

Primary Units

The six **primary** units of the SI system are shown in the table below:

Quantity	SI Unit	Dimension
length	metre, m	L
mass	kilogram, kg	M
time	second, s	T
temperature	Kelvin, K	Θ
current	*ampere, A*	*I*
luminosity	*candela*	*Cd*

In fluid mechanics we are generally only interested in the top four units from this table.

Notice how the term 'Dimension' of a unit has been introduced in this table. This is not a property of the individual units, rather it tells what the unit represents. For example a metre is a length which has a dimension L but also, an inch, a mile or a kilometre are all lengths so have dimension of L.

(The above notation uses the MLT system of dimensions, there are other ways of writing dimensions - we will see more about this in the section of the course on dimensional analysis.)

Derived Units

There are many **derived** units all obtained from combination of the above **primary** units. Those most used are shown in the table below:

Quantity	SI Unit		Dimension
velocity	m/s	ms^{-1}	LT^{-1}
acceleration	m/s^2	ms^{-2}	LT^{-2}
force	N kg m/s^2	kg ms^{-2}	M LT^{-2}
energy (or work)	Joule J N m, kg m^2/s^2	kg m^2s^{-2}	ML^2T^{-2}
power	Watt W N m/s kg m^2/s^3	Nms^{-1} kg m^2s^{-3}	ML^2T^{-3}
pressure (or stress)	Pascal P, N/m^2, $kg/m/s^2$	Nm^{-2} kg $m^{-1}s^{-2}$	$ML^{-1}T^{-2}$
density	kg/m^3	kg m^{-3}	ML^{-3}
specific weight	N/m^3 $kg/m^2/s^2$	kg $m^{-2}s^{-2}$	$ML^{-2}T^{-2}$
relative density	a ratio no units		1 no dimension
viscosity	N s/m^2 kg/m s	N sm^{-2} kg $m^{-1}s^{-1}$	M $L^{-1}T^{-1}$
surface tension	N/m kg $/s^2$	Nm^{-1} kg s^{-2}	MT^{-2}

The above units should be used at all times. Values in other units should NOT be used without first converting them into the appropriate SI unit. If you do not know what a particular unit means find out, else your guess will probably be wrong.
One very useful tip is to write down the units of any equation you are using. If at the end the units do not match you know you have made a mistake. For example is you have at the end of a calculation,

$$30 \text{ kg/m s} = 30 \text{ m}$$

you have certainly made a mistake - checking the units can often help find the mistake.

More on this subject will be seen later in the section on dimensional analysis and similarity.

Section 0. Examples: Units

1.

A water company wants to check that it will have sufficient water if there is a prolonged drought in the area. The region it covers is 500 square miles and the following consumption figures have been sent in by various different offices. There is sufficient information to calculate the amount of water available, but unfortunately it is in several different units.

Of the total area 100 000 acres is rural land and the rest urban. The density of the urban population is50 per square kilometre. The average toilet cistern is sized 200mm by 15in by 0.3m and on average each person uses this 3 time per day. The density of the rural population is 5 per square mile. Baths are taken twice a week by each person with the average volume of water in the bath being 6 gallons. Local industry uses 1000 m3 per week. Other uses are estimated as 5 gallons per person per day. A US air base in the region has given water use figures of 50 US gallons per person per day.

The average rain fall in 1in per month (28 days). In the urban area all of this goes to the river while in the rural area 10% goes to the river, 85% is lost (to the aquifer) and the rest goes to the one reservoir which supplies the region. This reservoir has an average surface area of 500 acres and is at a depth of 10 fathoms. 10% of this volume can be used in a month.

1. What is the total consumption of water per day in cubic meters?
2. If the reservoir was empty and no water could be taken from the river, would there be enough water if available if rain fall was only 10% of average?

Section 1: Fluids Mechanics and Fluid Properties

What is fluid mechanics? As its name suggests it is the branch of applied mechanics concerned with the statics and dynamics of fluids - both liquids and gases. The analysis of the behaviour of fluids is based on the fundamental laws of mechanics which relate continuity of mass and energy with force and momentum together with the familiar solid mechanics properties.

1. ## Objectives of this section
   - Define the nature of a fluid.
   - Show where fluid mechanics concepts are common with those of solid mechanics and indicate some fundamental areas of difference.
   - Introduce viscosity and show what are Newtonian and non-Newtonian fluids
   - Define the appropriate physical properties and show how these allow differentiation between solids and fluids as well as between liquids and gases.
2. ## Fluids

   There are two aspects of fluid mechanics which make it different to solid mechanics:

   1. The nature of a fluid is much different to that of a solid
   2. In fluids we usually deal with *continuous* streams of fluid without a beginning or end. In solids we only consider individual elements.

   We normally recognise three states of matter: solid; liquid and gas. However, liquid and gas are both fluids: in contrast to solids they lack the ability to resist deformation. Because a fluid cannot resist the deformation force, it moves, it *flows* under the action of the force. Its shape will change continuously as long as the force is applied. A solid can resist a deformation force while at rest, this force may cause some displacement but the solid does not continue to move indefinitely.

   The deformation is caused by *shearing* forces which act tangentially to a surface. Referring to the figure below, we see the force F acting tangentially on a rectangular (solid lined) element ABDC. This is a shearing force and produces the (dashed lined) rhombus element A'B'DC.

$$\sigma = \frac{\rho_{substance}}{\rho_{H_2O(at\,4^\circ c)}}$$

Shearing force, F, acting on a fluid element.

We can then say:

A Fluid is a substance which deforms continuously,
or flows, when subjected to shearing forces.

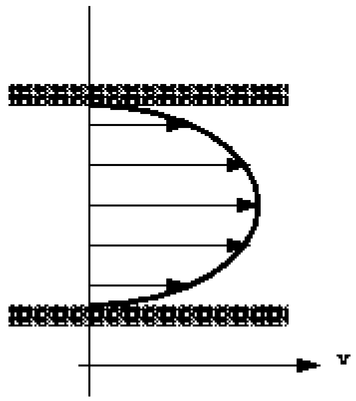
---

and conversely this definition implies the very important point that:

If a fluid is at rest there are no shearing forces acting.
All forces must be perpendicular to the planes which the are acting.

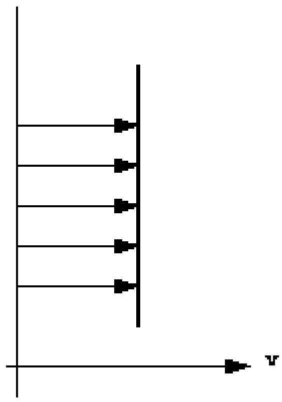
When a fluid is in motion shear stresses are developed if the particles of the fluid move relative to one another. When this happens adjacent particles have different velocities. If fluid velocity is the same at every point then there is no shear stress produced: the particles have zero *relative* velocity.

Consider the flow in a pipe in which water is flowing. At the pipe wall the velocity of the water will be zero. The velocity will increase as we move toward the centre of the pipe. This change in velocity across the direction of flow is known as velocity profile and shown graphically in the figure below:



Velocity profile in a pipe.

Because particles of fluid next to each other are moving with different velocities there **are** shear forces in the moving fluid i.e. shear forces are **normally** present in a moving fluid. On the other hand, if a fluid is a long way from the boundary and all the particles are travelling with the same velocity, the velocity profile would look something like this:

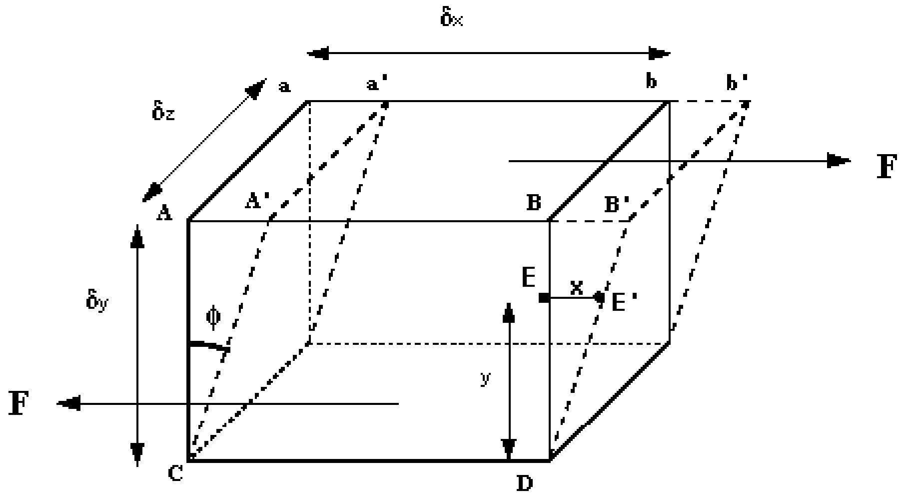


Velocity profile in uniform flow

and there will be no shear forces present as all particles have zero relative velocity. In practice we are concerned with flow past solid boundaries; aeroplanes, cars, pipe walls, river channels etc. and shear forces will be present.

3. Newton's Law of Viscosity

How can we make use of these observations? We can start by considering a 3d rectangular element of fluid, like that in the figure below.



Fluid element under a shear force

The shearing force F acts on the area on the top of the element. This area is given by $A = \delta z \times \delta x$. We can thus calculate the *shear stress* which is equal to force per unit area i.e.

$$\text{shear stress, } \tau = \frac{F}{A}$$

The deformation which this shear stress causes is measured by the size of the angle ϕ and is know as *shear strain*.

In a solid shear strain, ϕ, is constant for a fixed shear stress τ.
In a fluid ϕ increases for as long as τ is applied - the fluid flows.

It has been found experimentally that the *rate of shear stress* (shear stress per unit time, τ/time) is directly proportional to the shear stress.

If the particle at point E (in the above figure) moves under the shear stress to point E' and it takes time *t* to get there, it has moved the distance *x*. For small deformations we can write

shear strain $\phi = \frac{x}{y}$

$$\text{rate of shear strain} = \frac{\phi}{t} = \frac{x}{ty} = \frac{x}{t}\frac{1}{y} = \frac{u}{y}$$

where $\frac{x}{t} = u$ is the velocity of the particle at E.

Using the experimental result that shear stress is proportional to rate of shear strain then

$$\tau = \text{Constant} \times \frac{u}{y}$$

The term $\frac{u}{y}$ is the change in velocity with y, or the velocity gradient, and may be written in the differential form $\frac{du}{dy}$. The constant of proportionality is known as the dynamic viscosity, μ, of the fluid, giving

$$\tau = \mu \frac{du}{dy}$$

This is known as **Newton's law of viscosity**.

4. ## Fluids vs. Solids

In the above we have discussed the differences between the behaviour of solids and fluids under an applied force. Summarising, we have;

0. For a **solid** the strain is a function of the applied stress (providing that the elastic limit has not been reached). For a **fluid**, the rate of strain is proportional to the applied stress.

1. The strain in a **solid** is independent of the time over which the force is applied and (if the elastic limit is not reached) the deformation disappears when the force is removed. A **fluid** continues to flow for as long as the force is applied and will not recover its original form when the force is removed.

It is usually quite simple to classify substances as either solid or liquid. Some substances, however, (e.g. pitch or glass) appear solid under their own weight. Pitch will, although appearing solid at room temperature, deform and spread out over days - rather than the fraction of a second it would take water.

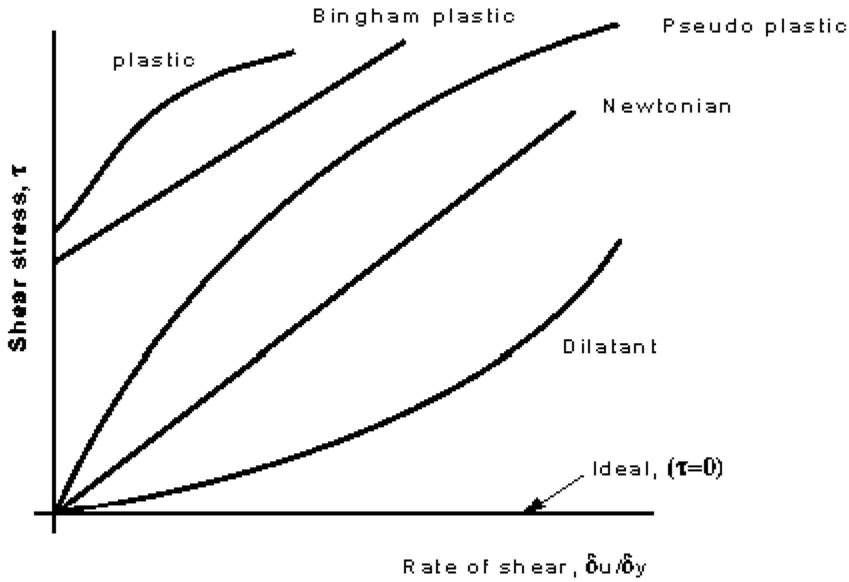
As you will have seen when looking at properties of solids, when the elastic limit is reached they seem to flow. They become plastic. They still do **not** meet the definition of true fluids as they will only flow after a certain minimum shear stress is attained.

5. ## Newtonian / Non-Newtonian Fluids

Even among fluids which are accepted as fluids there can be wide differences in behaviour under stress. Fluids obeying Newton's law where the value of μ is constant are known as **Newtonian** fluids. If μ is constant the shear stress is linearly dependent on velocity gradient. This is true for most common fluids.

Fluids in which the value of μ is not constant are known as **non-Newtonian** fluids. There are several categories of these, and they are outlined briefly below.

These categories are based on the relationship between shear stress and the velocity gradient (rate of shear strain) in the fluid. These relationships can be seen in the graph below for several categories



Shear stress vs. Rate of shear strain δu/δy

Each of these lines can be represented by the equation

$$\tau = A + B\left(\frac{\delta u}{\delta y}\right)^n$$

where A, B and n are constants. For Newtonian fluids A = 0, B = μ and n = 1.

Below are brief description of the physical properties of the several categories:

- *Plastic:* Shear stress must reach a certain minimum before flow commences.
- *Bingham plastic:* As with the plastic above a minimum shear stress must be achieved. With this classification n = 1. An example is sewage sludge.
- *Pseudo-plastic:* No minimum shear stress necessary and the viscosity decreases with rate of shear, e.g. colloidial substances like clay, milk and cement.
- *Dilatant substances;* Viscosity increases with rate of shear e.g. quicksand.
- *Thixotropic substances:* Viscosity decreases with length of time shear force is applied e.g. thixotropic jelly paints.
- *Rheopectic substances:* Viscosity increases with length of time shear force is applied
- *Viscoelastic materials:* Similar to Newtonian but if there is a sudden large change in shear they behave like plastic.

There is also one more - which is not real, it does not exist - known as the **ideal fluid**. This is a fluid which is assumed to have no viscosity. This is a useful concept when theoretical solutions are being considered - it does help achieve some practically useful solutions.

6. Liquids vs. Gasses

Although liquids and gasses behave in much the same way and share many similar characteristics, they also possess distinct characteristics of their own. Specifically

- A liquid is difficult to compress and often regarded as being incompressible.
  A gas is easily to compress and usually treated as such - it changes volume with pressure.
- A given mass of liquid occupies a given volume and will occupy the container it is in and form a free surface (if the container is of a larger volume).
  A gas has no fixed volume, it changes volume to expand to fill the containing vessel. It will completely fill the vessel so no free surface is formed.

7. Causes of Viscosity in Fluids

0. Viscosity in Gasses

The molecules of gasses are only weakly kept in position by molecular cohesion (as they are so far apart). As adjacent layers move by each other there is a continuous exchange of molecules. Molecules of a slower layer move to faster layers causing a drag, while molecules moving the other way exert an acceleration force. Mathematical considerations of this momentum exchange can lead to Newton law of viscosity.

If temperature of a gas increases the momentum exchange between layers will increase thus increasing viscosity.

Viscosity will also change with pressure - but under normal conditions this change is negligible in gasses.

1. Viscosity in Liquids

There is some molecular interchange between adjacent layers in liquids - but as the molecules are so much closer than in gasses the cohesive forces hold the molecules in place much more rigidly. This cohesion plays an important roll in the viscosity of liquids.

Increasing the temperature of a fluid reduces the cohesive forces and increases the molecular interchange. Reducing cohesive forces reduces shear stress, while increasing molecular interchange increases shear stress. Because of this complex interrelation the effect of temperature on viscosity has something of the form:

$$\mu_T = \mu_0 (1 + AT + BT)$$

where μ_T is the viscosity at temperature TC, and μ_0 is the viscosity at temperature 0C. A and B are constants for a particular fluid.

High pressure can also change the viscosity of a liquid. As pressure increases the relative movement of molecules requires more energy hence viscosity increases.

Properties of Fluids

The properties outlines below are general properties of fluids which are of interest in engineering. The symbol usually used to represent the property is specified together with some typical values in SI units for common fluids. Values under specific conditions (temperature, pressure etc.) can be readily found in many reference books. The dimensions of each unit is also give in the MLT system (see later in the section on dimensional analysis for more details about dimensions.)

1. Density

   The density of a substance is the quantity of matter contained in a unit volume of the substance. It can be expressed in three different ways.

   1. Mass Density

      Mass Density, ρ, is defined as the mass of substance per unit volume.

      Units: Kilograms per cubic metre, kg/m^3 (or kgm^{-3})

      Dimensions: ML^{-3}

      Typical values:

      Water = 1000 kgm^{-3}, Mercury = 13546 kgm^{-3} Air = 1.23 kgm^{-3}, Paraffin Oil = 800 kgm^{-3}.

      (at pressure =1.013 $\times 10^{-5}$ Nm^{-2} and Temperature = 288.15 K.)

   2. Specific Weight

Specific Weight ω, (sometimes , and sometimes known as *specific gravity*) is defined as the weight per unit volume.

or

The force exerted by gravity, g, upon a unit volume of the substance.

The Relationship between g and ω can be determined by Newton's 2nd Law, since

weight per unit volume = mass per unit volume g

$$\omega = \rho g$$

Units: Newton's per cubic metre, N/m^3 (or Nm^{-3})

Dimensions: $ML^{-2}T^{-2}$.

Typical values:

Water =9814 Nm^{-3}, Mercury = 132943 Nm^{-3}, Air =12.07 Nm^{-3}, Paraffin Oil =7851 Nm^{-3}

3. Relative Density

Relative Density, σ, is defined as the ratio of mass density of a substance to some standard mass density.

For solids and liquids this standard mass density is the maximum mass density for water (which occurs at 4°c) at atmospheric pressure.

$$\sigma = \frac{\rho_{substance}}{\rho_{H_2O(at 4^\circ c)}}$$

Units: None, since a ratio is a pure number.

Dimensions: 1.

Typical values: Water = 1, Mercury = 13.5, Paraffin Oil =0.8.

2. Viscosity

Viscosity, μ, is the property of a fluid, due to cohesion and interaction between molecules, which offers resistance to sheer deformation. Different fluids deform at different rates under the same shear stress. Fluid with a high viscosity such as syrup, deforms more slowly than fluid with a low viscosity such as water.

All fluids are viscous, "Newtonian Fluids" obey the linear relationship

given by Newton's law of viscosity. $\tau = \mu \frac{du}{dy}$, which we saw earlier.

where τ is the shear stress,

Units $N m^{-2}$; $kg\, m^{-1} s^{-2}$

Dimensions $ML^{-1}T^{-2}$.

$\frac{du}{dy}$ is the velocity gradient or rate of shear strain, and has

Units: $radians s^{-1}$,

Dimensions t^{-1}

μ is the "coefficient of dynamic viscosity" - see below.

1. ## Coefficient of Dynamic Viscosity

The Coefficient of Dynamic Viscosity, μ, is defined as the shear force, per unit area, (or shear stress τ), required to drag one layer of fluid with unit velocity past another layer a unit distance away.

$$\mu = \tau \Big/ \frac{du}{dy} = \frac{\text{Force}}{\text{Area}} \Big/ \frac{\text{Velocity}}{\text{Distance}} = \frac{\text{Force} \times \text{Time}}{\text{Area}} = \frac{\text{Mass}}{\text{Length} \times \text{Area}}$$

Units: Newton seconds per square metre, $N s m^{-2}$ or Kilograms per meter per second, $kgm^{-1} s^{-1}$.

(Although note that μ is often expressed in Poise, P, where 10 P = 1 $kgm^{-1} s^{-1}$.)

Typical values:

Water =1.14 $\times 10^{-3}\ kgm^{-1} s^{-1}$, Air =1.78 $\times 10^{-5}\ kgm^{-1} s^{-1}$, Mercury =1.552 $kgm^{-1} s^{-1}$,
Paraffin Oil =1.9 $kgm^{-1} s^{-1}$.

2. ## Kinematic Viscosity

Kinematic Viscosity, ν, is defined as the ratio of dynamic viscosity to mass density.

$$\nu = \frac{\mu}{\rho}$$

Units: square metres per second, $m^2 s^{-1}$

(Although note that ν is often expressed in Stokes, St, where 10^4 St = 1 $m^2 s^{-1}$.)

Dimensions: $L^2 T^{-1}$.

Typical values:

Water =1.14 $\times 10^{-6}\, m^2 s^{-1}$, Air =1.46 $\times 10^{-5}\, m^2 s^{-1}$, Mercury =1.145 $\times 10^{-4}\, m^2 s^{-1}$,
Paraffin Oil =2.375 $\times 10^{-3}\, m^2 s^{-1}$.

Fluid Properties Examples

1. Explain why the viscosity of a liquid decreases while that of a gas increases with a temperature rise.
   The following is a table of measurement for a fluid at constant temperature.
   Determine the dynamic viscosity of the fluid.
   [4.98 N/m^2]

du/dy (rad s^{-1})	0.00	0.20	0.40	0.60	0.80
τ (N m^{-2})	0.00	0.01	1.90	3.10	4.00

2. The density of an oil is 850 kg/m^3. Find its relative density and Kinematic viscosity if the dynamic viscosity is 5 x 10^{-3} kg/ms.
   [0.85, 1.47 x10^{-6} m^2/s]
3. The velocity distribution of a viscous liquid (dynamic viscosity μ = 0.9 Ns/m^2) flowing over a fixed plate is given by u = 0.68y - y^2 (u is velocity in m/s and y is the distance from the plate in m).
   What are the shear stresses at the plate surface and at y=0.34m?
   [0.612 N/m^2, 0]
4. 5.6m^3 of oil weighs 46 800 N. Find its mass density, ρ, and relative density, γ.
   [852 kg/m^3, 0.852]
5. From table of fluid properties the viscosity of water is given as 0.01008 poises.

What is this value in Ns/m^2 and Pa s units?
[0.001008 Ns/m^2]

6. In a fluid the velocity measured at a distance of 75mm from the boundary is 1.125m/s. The fluid has absolute viscosity 0.048 Pa s and relative density 0.913. What is the velocity gradient and shear stress at the boundary assuming a linear velocity distribution.
[15 s^{-1}, 0.720 Pa]

Fluid Properties Examples

Q1 Q2 Q3 Q4 Q5 Q6

1. Explain why the viscosity of a liquid decreases while that of a gas increases with a temperature rise.
The following is a table of measurement for a fluid at constant temperature.
Determine the dynamic viscosity of the fluid.

du/dy (s^{-1})	*0.0*	*0.2*	*0.4*	*0.6*	*0.8*
τ (N m^{-2})	0.0	1.0	1.9	3.1	4.0

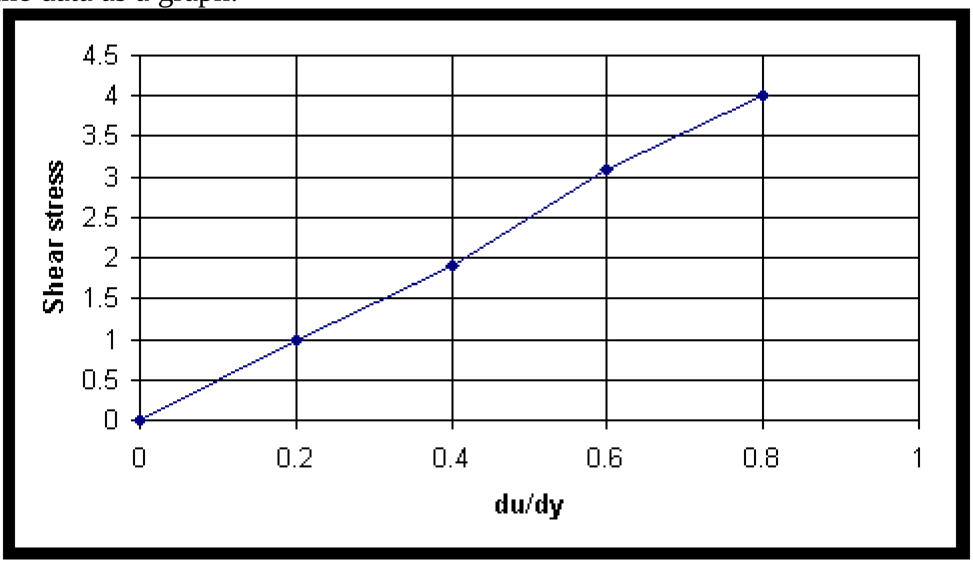
Using Newton's law of viscocity

$$\tau = \mu \frac{\partial u}{\partial y}$$

where μ is the viscosity. So viscosity is the gradient of a graph of shear stress against vellocity gradient of the above data, or

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}}$$

Plot the data as a graph:



Calculate the gradient for each section of the line

du/dy (s^{-1})	*0.0*	*0.2*	*0.4*	*0.6*	*0.8*
τ (N m^{-2})	0.0	1.0	1.9	3.1	4.0
Gradient	-	5.0	4.75	5.17	5.0

Thus the mean gradient = viscosity = 4.98 N s / m^2

---

2. The density of an oil is 850 kg/m^3. Find its relative density and Kinematic viscosity if the dynamic viscosity is 5×10^{-3} kg/ms.

ρ_{oil} = 850 kg/m^3 ρ_{water} = 1000 kg/m^3

γ_{oil} = 850 / 1000 = 0.85

Dynamic viscosity = $\mu = 5\times 10^{-3}$ kg/ms

Kinematic viscosity = $\nu = \mu / \rho$

$$\nu = \frac{\mu}{\rho} = \frac{5\times10^{-3}}{1000} = 5\times10^{-6}\ m^2 s^{-1}$$

---

3. The velocity distribution of a viscous liquid (dynamic viscosity μ = 0.9 Ns/m^2) flowing over a fixed plate is given by u = 0.68y - y^2 (u is velocity in m/s and y is the distance from the plate in m).
What are the shear stresses at the plate surface and at y=0.34m?

$$u = 0.68y - y^2$$
$$\frac{\partial u}{\partial y} = 0.68 - 2y$$

At the plate face y = 0m,

$$\frac{\partial u}{\partial y} = 0.68$$

Calculate the shear stress at the plate face

$$\tau = \mu \frac{\partial u}{\partial y} = 0.9 \times 0.68 = 0.612\, N/m^2$$

At y = 0.34m,

$$\frac{\partial u}{\partial y} = 0.68 - 2 \times 0.34 = 0.0$$

As the velocity gradient is zero at y=0.34 then the shear stress must also be zero.

---

4. 5.6m^3 of oil weighs 46 800 N. Find its mass density, ρ, and relative density, γ.

   Weight 46 800 = mg

   Mass m = 46 800 / 9.81 = 4770.6 kg

   Mass density ρ = Mass / volume = 4770.6 / 5.6 = 852 kg/m^3

   Relative density $\gamma = \frac{\rho}{\rho_{water}} = \frac{852}{1000} = 0.852$

---

5. From table of fluid properties the viscosity of water is given as 0.01008 poises.
   What is this value in Ns/m^2 and Pa s units?

   μ = 0.01008 poise

   1 poise = 0.1 Pa s = 0.1 Ns/m^2

   μ = 0.001008 Pa s = 0.001008 Ns/m^2

---

6. In a fluid the velocity measured at a distance of 75mm from the boundary is 1.125m/s. The fluid has absolute viscosity 0.048 Pa s and relative density 0.913. What is the velocity gradient and shear stress at the boundary assuming a linear velocity distribution.

   μ = 0.048 Pa s

   γ = 0.913

$$\frac{\partial u}{\partial y} = \frac{1.125}{0.075} = 15 s^{-1}$$

$$\tau = \mu \frac{\partial u}{\partial y}$$

$$= 0.048 \times 15 = 0.720 Pa s$$

SECTION 2: Forces in Static Fluids

This section will study the forces acting on or generated by fluids at rest.

Objectives

- Introduce the concept of pressure;
- Prove it has a unique value at any particular elevation;
- Show how it varies with depth according to the hydrostatic equation and
- Show how pressure can be expressed in terms of *head* of fluid.

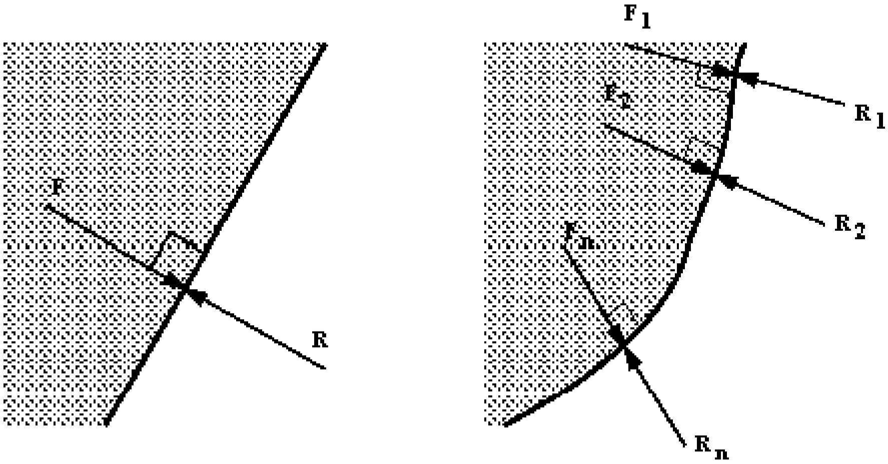
This understanding of pressure will then be used to demonstrate methods of pressure measurement that will be useful later with fluid in motion and also to analyse the forces on submerges surface/structures.

1. Fluids statics

The general rules of statics (as applied in solid mechanics) apply to fluids at rest. From earlier we know that:

- a static fluid can have **no shearing force** acting on it, and that

- any force between the fluid and the boundary must be acting at right angles to the boundary.



Pressure force normal to the boundary

Note that this statement is also true for curved surfaces, in this case the force acting at any point is normal to the surface at that point. The statement is also true for any imaginary plane in a static fluid. We use this fact in our analysis by considering elements of fluid bounded by imaginary planes.

We also know that:

- For an element of fluid at rest, the element will be in equilibrium - the sum of the components of forces in any direction will be zero.
- The sum of the moments of forces on the element about any point must also be zero.

It is common to test equilibrium by resolving forces along three mutually perpendicular axes and also by taking moments in three mutually perpendicular planes an to equate these to zero.

2. Pressure

As mentioned above a fluid will exert a normal force on any boundary it is in contact with. Since these boundaries may be large and the force may differ from place to place it is convenient to work in terms of pressure, *p*, which is the force per unit area.

If the force exerted on each unit area of a boundary is the same, the pressure is said to be *uniform*.

$$\text{pressure} = \frac{\text{Force}}{\text{Area over which the force is applied}}$$

$$p = \frac{F}{A}$$

Units: Newton's per square metre, $N m^{-2}$, $kg m^{-1} s^{-2}$.

(The same unit is also known as a Pascal, *Pa*, i.e. 1*Pa* = 1 $N m^{-2}$)

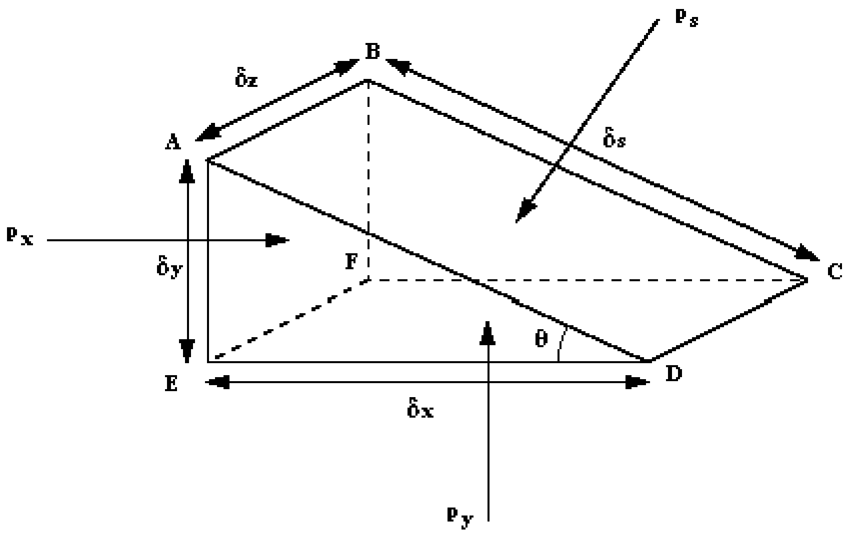
(Also frequently used is the alternative SI unit the *bar,* where $1 bar = 10^5 N m^{-2}$)

Dimensions: $M L^{-1} T^{-2}$.

3. Pascal's Law for Pressure At A Point

(Proof that pressure acts equally in all directions.)

By considering a small element of fluid in the form of a triangular prism which contains a point P, we can establish a relationship between the three pressures p_x in the x direction, p_y in the y direction and p_s in the direction normal to the sloping face.



Triangular prismatic element of fluid

The fluid is a rest, so we know there are no shearing forces, and we know that all force are acting at right angles to the surfaces .i.e.

p_s acts perpendicular to surface ABCD,

p_x acts perpendicular to surface ABFE and

p_y acts perpendicular to surface FECD.

And, as the fluid is at rest, in equilibrium, the sum of the forces in any direction is zero.

Summing forces in the x-direction:

Force due to p_x,

$$F_{x_x} = p_x \times Area_{ABFE} = p_x \delta x \, \delta y$$

Component of force in the x-direction due to p_s,

$$\begin{aligned} F_{x_s} &= -p_s \times Area_{ABCD} \times \sin\theta \\ &= -p_s \delta s \,\delta z \frac{\delta y}{\delta s} \\ &= -p_s \delta y \,\delta z \end{aligned}$$

($\sin\theta = \delta y / \delta s$)

Component of force in x-direction due to p_y,

$$F_{x_y} = 0$$

To be at rest (in equilibrium)

$$\begin{aligned} F_{x_x} + F_{x_s} + F_{x_y} &= 0 \\ p_x \delta x \,\delta y + \left(-p_s \delta y \,\delta z\right) &= 0 \\ p_x &= p_s \end{aligned}$$

Similarly, summing forces in the y-direction. Force due to p_y,

$$F_{y_y} = p_y \times Area_{EFCD} = p_y \delta x \,\delta z$$

Component of force due to p_s,

$$\begin{aligned} F_{y_s} &= -p_s \times Area_{ABCD} \times \cos\theta \\ &= -p_s \delta s \,\delta z \frac{\delta x}{\delta s} \\ &= -p_s \delta x \,\delta z \end{aligned}$$

($\cos\theta = \delta x / \delta s$)

Component of force due to p_x,

$$F_{y_x} = 0$$

Force due to gravity,

$$\begin{aligned} \text{weight} &= -\text{specific weight} \times \text{volume of element} \\ &= -\rho g \times \frac{1}{2} \delta x \,\delta y \,\delta z \end{aligned}$$

To be at rest (in equilibrium)

$$F_{y_y} + F_{y_s} + F_{y_x} + \text{ weight } = 0$$

$$p_y \delta x \delta y + \left(-p_s \delta x \delta z\right) + \left(-\rho g \frac{1}{2} \delta x \delta y \delta z\right) = 0$$

The element is small i.e. δx , δy and δz are small, and so $\delta x \delta y \delta z$ is <u>very</u> small and considered negligible, hence

$$p_y = p_s$$

thus

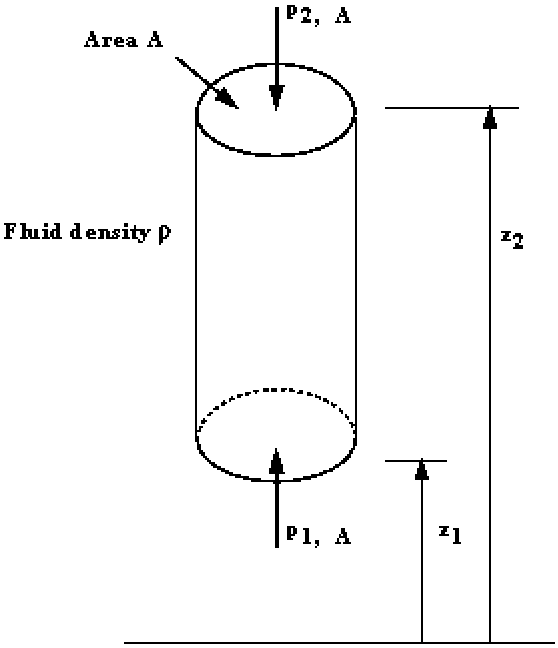
$$p_x = p_y = p_s$$

Considering the prismatic element again, p_s is the pressure on a plane at any angle θ, the x, y and z directions could be any orientation. The element is so small that it can be considered a point so the derived expression $p_x = p_y = p_s$. indicates that pressure at any point is the same in all directions. (The proof may be extended to include the z axis).

Pressure at any point is the same in all directions.
This is known as **Pascal's Law** and applies to fluids at rest.

---

4. Variation Of Pressure Vertically In A Fluid Under Gravity



Vertical elemental cylinder of fluid

In the above figure we can see an element of fluid which is a vertical column of constant cross sectional area, A, surrounded by the same fluid of mass density ρ. The pressure at the bottom of the cylinder is p_1 at level z_1, and at the top is p_2 at level z_2. The fluid is at rest and in equilibrium so all the forces in the vertical direction sum to zero. i.e. we have

$$
\begin{aligned}
\text{Force due to } p_1 \text{ on A (upward)} &= p_1 A \\
\text{Force due to } p_2 \text{ on A (downward)} &= p_2 A \\
\text{Force due to weight of element (downward)} &= mg \\
&= \text{mass density} \times \text{volume} = \rho g A(z_2 - z_1)
\end{aligned}
$$

Taking upward as positive, in equilibrium we have

$$p_1 A - p_2 A - \rho g A(z_2 - z_1) = 0$$

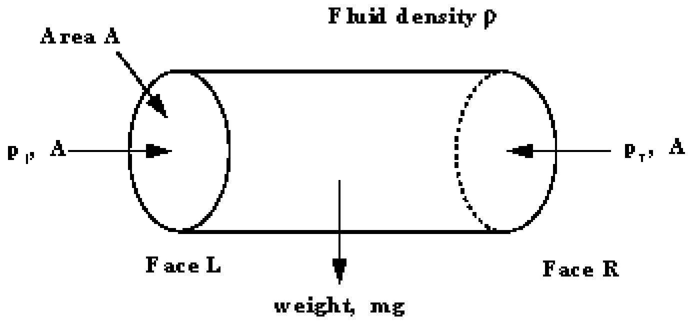
$$p_2 - p_1 = -\rho g(z_2 - z_1)$$

Thus in a fluid under gravity, pressure decreases with increase in height $z = (z_2 - z_1)$.

---

5. Equality Of Pressure At The Same Level In A Static Fluid

Consider the horizontal cylindrical element of fluid in the figure below, with cross-sectional area A, in a fluid of density ρ, pressure p_1 at the left hand end and pressure p_2 at the right hand end.



Horizontal elemental cylinder of fluid

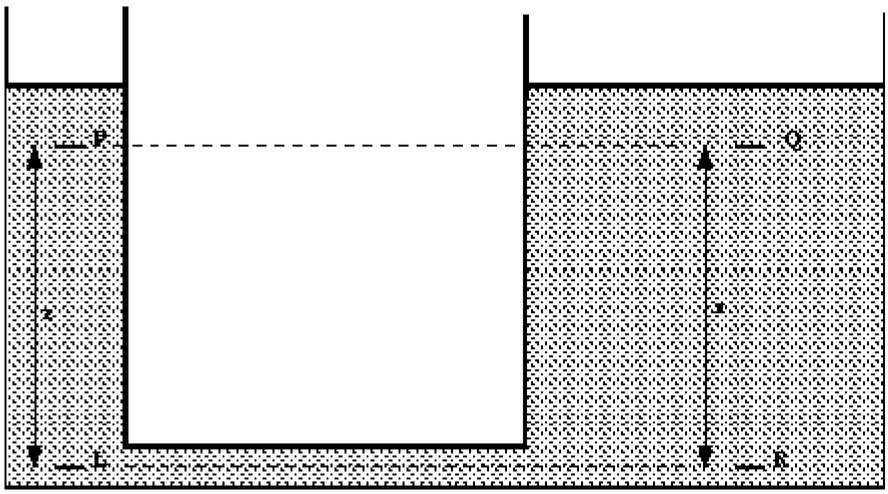
The fluid is at equilibrium so the sum of the forces acting in the x direction is zero.

$$p_l A = p_r A$$

$$p_l = p_r$$

Pressure in the horizontal direction is constant.

This result is the same for any *continuous* fluid. It is still true for two connected tanks which appear not to have any direct connection, for example consider the tank in the figure below.



Two tanks of different cross-section connected by a pipe

We have shown above that $p_l = p_r$ and from the equation for a vertical pressure change we have

$$p_l = p_p + \rho g z$$

and

$$p_r = p_q + \rho g z$$

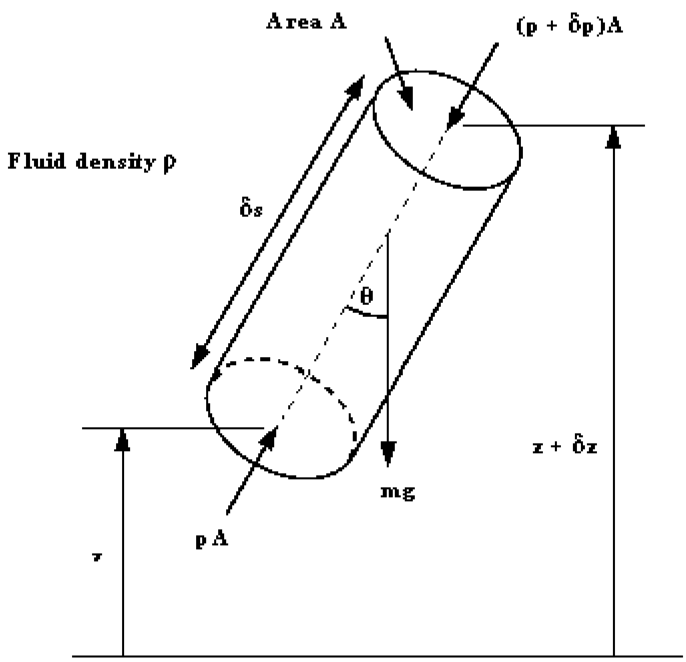
so

$$p_p + \rho g z = p_q + \rho g z$$
$$p_p = p_q$$

This shows that the pressures at the two equal levels, P and Q are the same.

6. General Equation For Variation Of Pressure In A Static Fluid

Here we show how the above observations for vertical and horizontal elements of fluids can be generalised for an element of any orientation.



A cylindrical element of fluid at an arbitrary orientation.

Consider the cylindrical element of fluid in the figure above, inclined at an angle θ to the vertical, length δs, cross-sectional area A in a static fluid of mass density ρ. The pressure at the end with height z is p and at the end of height $z + \delta z$ is $p + \delta p$.

The forces acting on the element are

$$\begin{aligned}
&pA \quad \text{acting at right - angles to the end of the face at } z \\
&(p + \delta p)A \quad \text{acting at right - angles to the end of the face at } z + \delta z \\
&mg = \text{the weight of the element acting vertically down} \\
&\quad = \text{mass density} \times \text{volume} \times \text{gravity} \\
&\quad = \rho A \delta s g
\end{aligned}$$

There are also forces from the surrounding fluid acting normal to these sides of the element.

For equilibrium of the element the resultant of forces in any direction is zero.

Resolving the forces in the direction along the central axis gives

$$pA - (p + \delta p)A - \rho g A \delta s \cos\theta = 0$$

$$\delta p = -\rho g \delta s \cos\theta$$

$$\frac{\delta p}{\delta s} = -\rho g \cos\theta$$

Or in the differential form

$$\frac{dp}{ds} = -\rho g \cos\theta$$

If $\theta = 90^\circ$ then *s* is in the *x* or *y* directions, (i.e. horizontal),so

$$\left(\frac{dp}{ds}\right)_{\theta=90^\circ} = \frac{dp}{dx} = \frac{dp}{dy} = 0$$

Confirming that pressure on any horizontal plane is zero.

If $\theta = 0^\circ$ then *s* is in the *z* direction (vertical) so

$$\left(\frac{dp}{ds}\right)_{\theta=0^\circ} = \frac{dp}{dz} = -\rho g$$

Confirming the result

$$\frac{p_2 - p_1}{z_2 - z_1} = -\rho g$$

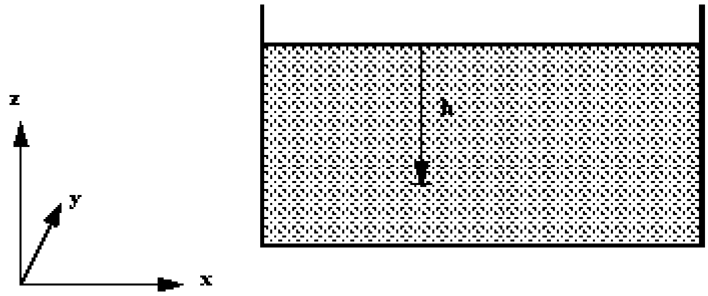
$$p_2 - p_1 = -\rho g (z_2 - z_1)$$

7. Pressure And Head

In a static fluid of constant density we have the relationship $\frac{dp}{dz} = -\rho g$, as shown above. This can be integrated to give

$$p = -\rho g z + \text{constant}$$

In a liquid with a free surface the pressure at any depth *z* measured from the free surface so that *z* = -*h* (see the figure below)



Fluid head measurement in a tank.

This gives the pressure

$$p = \rho g h + \text{constant}$$

At the surface of fluids we are normally concerned with, the pressure is the atmospheric pressure, $p_{\text{atmospheri}}$. So

$$p = \rho g h + p_{\text{atmospheri}}$$

As we live constantly under the pressure of the atmosphere, and everything else exists under this pressure, it is convenient (and often done) to take atmospheric pressure as the datum. So we quote pressure as above or below atmospheric.

Pressure quoted in this way is known as gauge pressure i.e.

Gauge pressure is

$$p_{\text{gauge}} = \rho g h$$

The lower limit of any pressure is zero - that is the pressure in a perfect vacuum. Pressure measured above this datum is known as absolute pressure i.e.

Absolute pressure is

$$p_{\text{absolute}} = \rho g h + p_{\text{atmospheric}}$$
$$\text{Absolute pressure} = \text{Gauge prssure} + \text{Atmospheric pressure}$$

As *g* is (approximately) constant, the gauge pressure can be given by stating the vertical height of any fluid of density ρ which is equal to this pressure.

$$p = \rho g h$$

This vertical height is known as **head** of fluid.

Note: If pressure is quoted in *head*, the density of the fluid *must* also be given.

Example:

We can quote a pressure of $500\,K\,N m^{-2}$ in terms of the height of a column of water of density, $\rho = 1000 kg m^{-3}$. Using $p = \rho g h$,

$$h = \frac{p}{\rho g} = \frac{500 \times 10^3}{1000 \times 9.81} = 50.95 m \text{ of water}$$

And in terms of Mercury with density, $\rho = 13.6 \times 10^3 kg m^{-3}$.

$$h = \frac{500 \times 10^3}{13.6 \times 10^3 \times 9.81} = 3.75 m \text{ of Mercury}$$

Pressure Measurement By Manometer

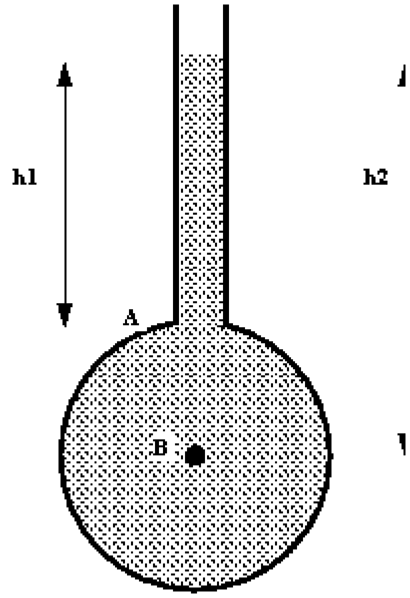
The relationship between pressure and head is used to measure pressure with a manometer (also know as a liquid gauge).

Objective:

- To demonstrate the analysis and use of various types of manometers for pressure measurement.

1. The Piezometer Tube Manometer

The simplest manometer is a tube, open at the top, which is attached to the top of a vessel containing liquid at a pressure (higher than atmospheric) to be measured. An example can be seen in the figure below. This simple device is known as a *Piezometer tube*. As the tube is open to the atmosphere the pressure measured is relative to atmospheric so is **gauge pressure**.



A simple piezometer tube manometer

$$\text{pressure at A} = \text{pressure due to column of liquid above A}$$

$$p_A = \rho g h_1$$

$$\text{pressure at B} = \text{pressure due to column of liquid above B}$$

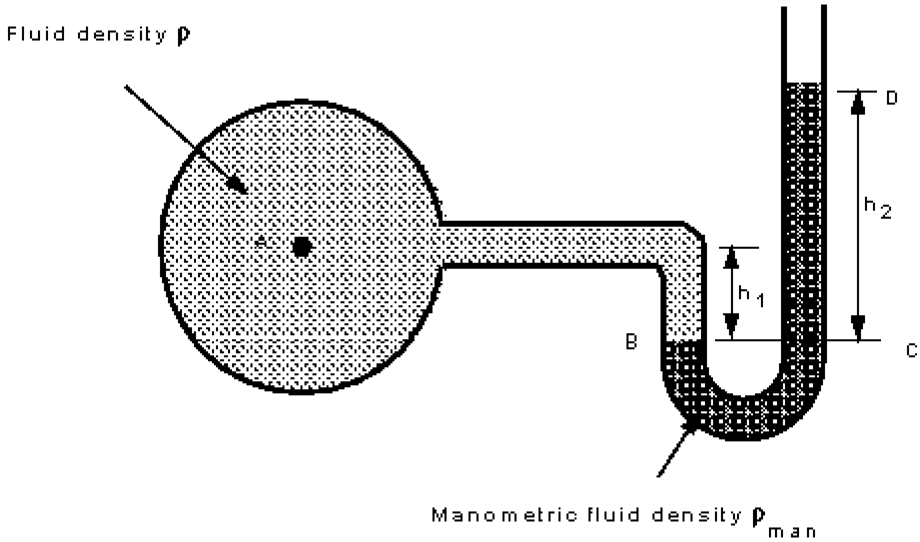
$$p_B = \rho g h_2$$

This method can only be used for liquids (i.e. **not** for gases) and only when the liquid height is convenient to measure. It must not be too small or too large and pressure changes must be detectable.

2. The "U"-Tube Manometer

Using a "U"-Tube enables the pressure of both liquids and gases to be measured with the same instrument. The "U" is connected as in the figure below and filled with a fluid called the *manometric fluid*. The fluid whose pressure is being measured should have a mass density less than that of the

manometric fluid and the two fluids should not be able to mix readily - that is, they must be immiscible.



A "U"-Tube manometer

Pressure in a continuous static fluid is the same at any horizontal level so,

$$\text{pressure at B} = \text{pressure at C}$$
$$p_B = p_C$$

For the **left hand** arm

$$\text{pressure at B} = \text{pressure at A} + \text{pressure due to height } h_1 \text{ of fluid being measured}$$
$$p_B = p_A + \rho g h_1$$

For the **right hand** arm

$$\text{pressure at C} = \text{pressure at D} + \text{pressure due to height } h_2 \text{ of manometric fluid}$$
$$p_C = p_{\text{Amospheric}} + \rho_{\text{man}} g h_2$$

As we are measuring *gauge pressure* we can subtract $p_{\text{Atmospheri}}$ giving

$$p_B = p_C$$

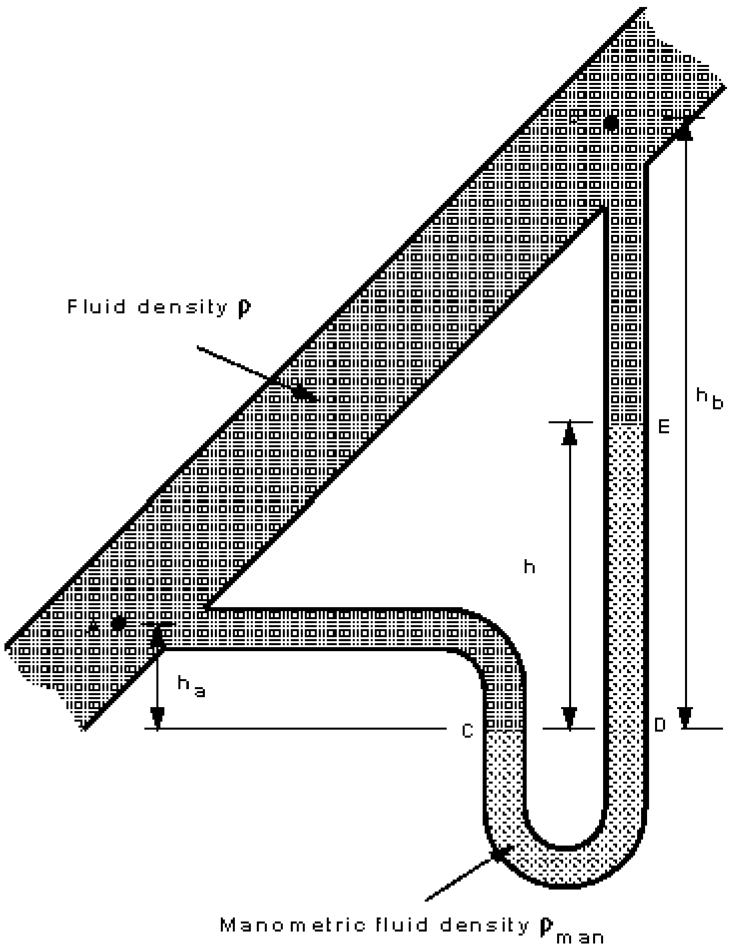
$$p_A = \rho_{\text{man}} g h_2 - \rho g h_1$$

If the fluid being measured is a gas, the density will probably be very low in comparison to the density of the manometric fluid i.e. $\rho_{man} >> \rho$. In this case the term $\rho g h_1$ can be neglected, and the gauge pressure give by

$$p_A = \rho_{\text{man}} g h_2$$

3. Measurement Of Pressure Difference Using a "U"-Tube Manometer.

If the "U"-tube manometer is connected to a pressurised vessel at two points the *pressure difference* between these two points can be measured.



Pressure difference measurement by the "U"-Tube manometer

If the manometer is arranged as in the figure above, then

$$\text{pressure at C} = \text{pressure at C}$$

$$p_C = p_D$$

$$p_C = p_A + \rho g h_a$$

$$p_D = p_B + \rho g\left(h_b - h\right) + \rho_{man} g h$$

$$p_A + \rho g h_a = p_B + \rho g\left(h_b - h\right) + \rho_{man} g h$$

Giving the pressure difference

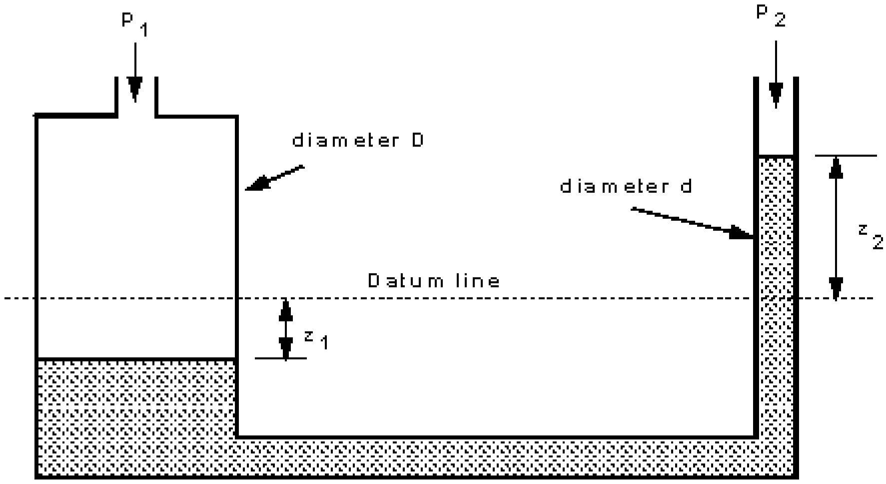
$$p_A - p_B = \rho g\left(h_b - h_a\right) + \left(\rho_{man} - \rho\right) g h$$

Again, if the fluid whose pressure difference is being measured is a gas and $\rho_{man} >> \rho$, then the terms involving ρ can be neglected, so

$$p_A - p_B = \rho_{man} g h$$

4. Advances to the "U" tube manometer.

The "U"-tube manometer has the disadvantage that the change in height of the liquid in both sides must be read. This can be avoided by making the diameter of one side very large compared to the other. In this case the side with the large area moves very little when the small area side move considerably more.



Assume the manometer is arranged as above to measure the pressure difference of a gas of (negligible density) and that pressure difference is $p_1 - p_2$. If the datum line indicates the level of the manometric fluid when the pressure difference is zero and the height differences when pressure is applied is as shown, the volume of liquid transferred from the left side to the right $= z_2 \times \left(\pi d^2 / 4\right)$

And the fall in level of the left side is

$$\begin{aligned} z_1 &= \frac{\text{Volume moved}}{\text{Area of left side}} \\ &= \frac{z_2\left(\pi d^2 / 4\right)}{\pi D^2 / 4} \\ &= z_2\left(\frac{d}{D}\right)^2 \end{aligned}$$

We know from the theory of the "U" tube manometer that the height different in the two columns gives the pressure difference so

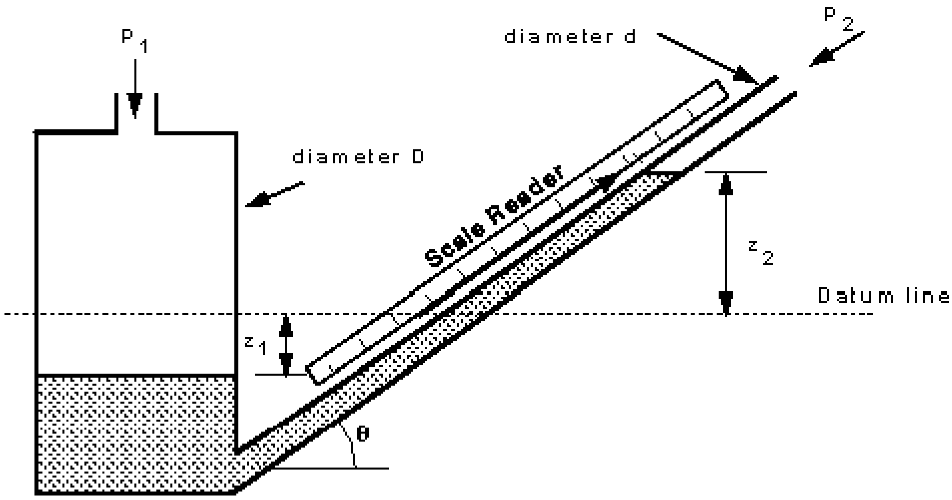
$$p_1 - p_2 = \rho g \left[z_2 + z_2 \left(\frac{d}{D} \right)^2 \right]$$
$$= \rho g z_2 \left[1 + \left(\frac{d}{D} \right)^2 \right]$$

Clearly if D is very much larger than *d* then $(d/D)^2$ is very small so

$$p_1 - p_2 = \rho g z_2$$

So only one reading need be taken to measure the pressure difference.

If the pressure to be measured is very small then tilting the arm provides a convenient way of obtaining a larger (more easily read) movement of the manometer. The above arrangement with a tilted arm is shown in the figure below.



Tilted manometer.

The pressure difference is still given by the height change of the manometric fluid but by placing the scale along the line of the tilted arm and taking this reading large movements will be observed. The pressure difference is then given by

$$p_1 - p_2 = \rho g z_2$$
$$= \rho g x \sin \theta$$

The sensitivity to pressure change can be increased further by a greater inclination of the manometer arm, alternatively the density of the manometric fluid may be changed.

5. Choice Of Manometer

Care must be taken when attaching the manometer to vessel, no burrs must be present around this joint. Burrs would alter the flow causing local pressure variations to affect the measurement.

Some disadvantages of manometers:

- Slow response - only really useful for very slowly varying pressures - no use at all for fluctuating pressures;
- For the "U" tube manometer two measurements must be taken simultaneously to get the h value. This may be avoided by using a tube with a much larger cross-sectional area on one side of the manometer than the other;
- It is often difficult to measure small variations in pressure - a different manometric fluid may be required - alternatively a sloping manometer may be employed; It cannot be used for very large pressures unless several manometers are connected in series;
- For very accurate work the temperature and relationship between temperature and ρ must be known;

Some advantages of manometers:

- They are very simple.
- No calibration is required - the pressure can be calculated from first principles.

Forces on Submerged Surfaces in Static Fluids

We have seen the following features of statics fluids

- Hydrostatic vertical pressure distribution
- Pressures at any equal depths in a continuous fluid are equal
- Pressure at a point acts equally in all directions (Pascal's law).
- Forces from a fluid on a boundary acts at right angles to that boundary.

Objectives:

We will use these to analyse and obtain expressions for the forces on submerged surfaces. In doing this it should also be clear the difference between:

- Pressure which is a scalar quantity whose value is equal in all directions and,
- Force, which is a vector quantity having both magnitude and direction.

1. Fluid pressure on a surface

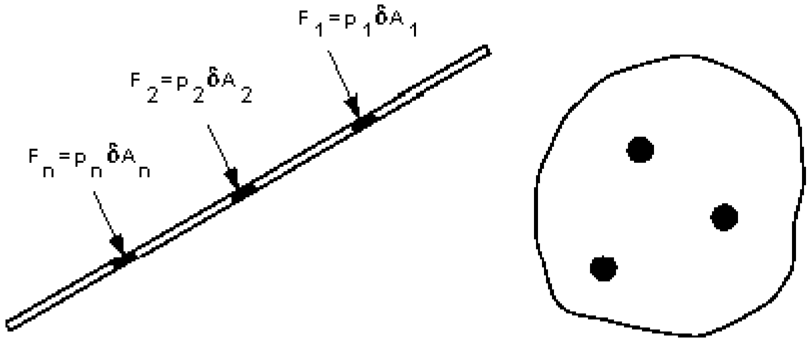
Pressure is defined as force per unit area. If a pressure *p* acts on a small area δA then the force exerted on that area will be

$$F = p\,\delta A$$

Since the fluid is at rest the force will act at right-angles to the surface.

General submerged plane

Consider the plane surface shown in the figure below. The total area is made up of many elemental areas. The force on each elemental area is always normal to the surface but, in general, each force is of different magnitude as the pressure usually varies.



We can find the total or ***resultant*** force, *R*, on the plane by summing up all of the forces on the small elements i.e.

$$R = p_1\delta A_1 + p_2\delta A_2 + \ldots\ldots p_n\delta A_n = \sum p\,\delta A$$

This resultant force will act through the centre of pressure, hence we can say

---

If the surface is a **plane** the force can be represented by one single **resultant force**, acting at right-angles to the plane through the **centre of pressure**.

Horizontal submerged plane

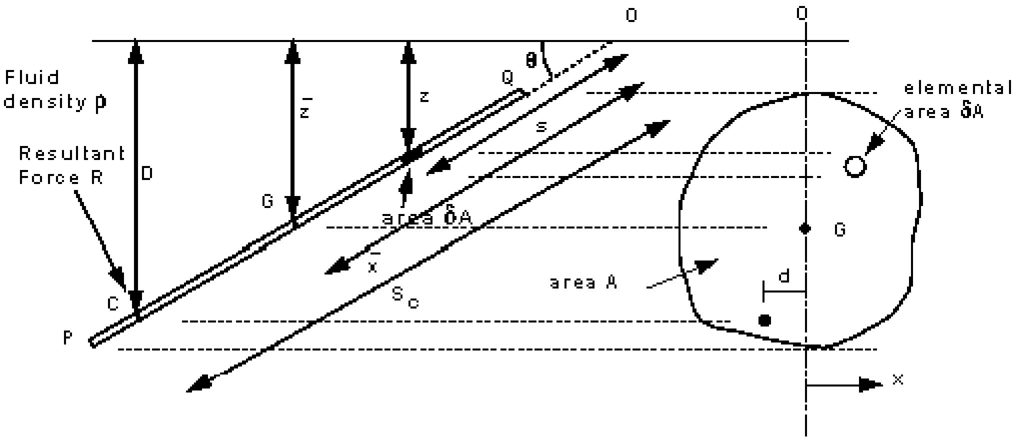
For a horizontal plane submerged in a liquid (or a plane experiencing uniform pressure over its surface), the pressure, *p*, will be equal at all points of the surface. Thus the resultant force will be given by

$$R = \text{pressure} \times \text{area of plane}$$
$$R = pA$$

Curved submerged surface

If the surface is curved, each elemental force will be a different magnitude and in different direction but still normal to the surface of that element. The resultant force can be found by resolving all forces into orthogonal co-ordinate directions to obtain its magnitude and direction. This will **always** be less than the sum of the individual forces, $\sum p\,\delta A$.

2. Resultant Force and Centre of Pressure on a submerged plane surface in a liquid.



This plane surface is totally submerged in a liquid of density ρ and inclined at an angle of θ to the horizontal. Taking pressure as zero at the surface and measuring down from the surface, the pressure on an element δA, submerged a distance *z*, is given by

$$p = \rho g z$$

and therefore the force on the element is

$$F = p\,\delta A = \rho g z\,\delta A$$

The resultant force can be found by summing all of these forces i.e.

$$R = \rho g \sum z\,\delta A$$

(assuming ρ and *g* as constant).

The term $\sum z\,\delta A$ is known as the *1st Moment of Area* of the plane PQ about the free surface. It is equal to $A\bar{z}$ i.e.

$$\sum z\,\delta A = A\bar{z}$$
$$= 1^{st} \text{ moment of area about the line of the free surface}$$

where *A* is the area of the plane and $\bar{z}$ is the depth (distance from the free surface) to the centroid, G. This can also be written in terms of distance from point O (as $\bar{z} = \bar{x}\sin\theta$)

$$\sum z\,\delta A = A\bar{x}\sin\theta$$
$$= 1^{st} \text{ Moment of area about a line through O} \times \sin\theta$$

Thus:

The resultant force on a plane

$$R = \rho g A \bar{z}$$
$$= \rho g A \bar{x} \sin\theta$$

This resultant force acts at right angles to the plane through the centre of pressure, C, at a depth D. The moment of *R* about any point will be equal to the sum of the moments of the forces on all the elements δA of the plane about the same point. We use this to find the position of the centre of pressure.

It is convenient to take moments about the point where a projection of the plane passes through the surface, point O in the figure.

$$\text{Moment of } R \text{ about O} = \text{Sum of moments of force on all elements of } \delta A \text{ about O}$$

We can calculate the force on each elemental area:

$$\text{Force on } \delta A = \rho g z \delta A$$
$$= \rho g s \sin\theta \, \delta A$$

And the moment of this force is:

$$\text{Moment of Force on } \delta A \text{ about O} = \rho g s \sin\theta \, \delta A \times s$$
$$= \rho g \sin\theta \, \delta A s^2$$

ρ, g and θ are the same for each element, so the total moment is

$$\text{Sum of moments of forces on all elemets of } \delta A \text{ about O} = \rho g \sin\theta \sum s^2 \delta A$$

We know the resultant force from above $R = \rho g A \bar{x} \sin\theta$, which acts through the centre of pressure at C, so

$$\text{Moment of } R \text{ about O} = \rho g A \bar{x} \sin\theta \, S_c$$

Equating gives,

$$\rho g A \bar{x} \sin\theta \, S_c = \rho g \sin\theta \sum s^2 \delta A$$

Thus the position of the centre of pressure along the plane measure from the point O is:

$$S_c = \frac{\sum s^2 \delta A}{A \bar{x}}$$

It look a rather difficult formula to calculate - particularly the summation term. Fortunately this term is known as the *2nd Moment of Area* , I_o , of the plane about the axis through O and it can be easily calculated for many common shapes. So, we know:

$$\text{2nd moment of area about O} = I_o = \sum s^2 \delta A$$

And as we have also seen that $A\bar{x} =$ 1st Moment of area about a line through O,

---

Thus the position of the centre of pressure along the plane measure from the point O is:

$$S_c = \frac{2^{nd}\ \text{Moment of area about a line through O}}{1^{st}\ \text{Moment of area about a line through O}}$$

---

and depth to the centre of pressure is

$$D = S_c \sin\theta$$

How do you calculate the 2nd moment of area?

To calculate the 2nd moment of area of a plane about an axis through O, we use the *parallel axis theorem* together with values of the 2nd moment of area about an axis though the centroid of the shape obtained from tables of geometric properties.

The *parallel axis theorem* can be written

$$I_o = I_{GG} + A\bar{x}^2$$

---

where I_{GG} is the 2nd moment of area about an axis though the centroid G of the plane.

Using this we get the following expressions for the position of the centre of pressure

---

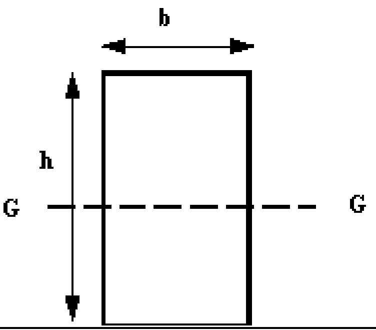
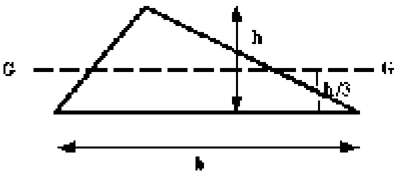
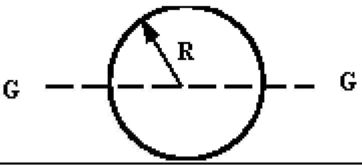
$$S_c = \frac{I_{GG}}{A\bar{x}} + \bar{x}$$

$$D = \sin\theta\left(\frac{I_{GG}}{A\bar{x}} + \bar{x}\right)$$

(In the examination the parallel axis theorem and the I_{GG} will be given)

The second moment of area of some common shapes.

The table blow given some examples of the 2nd moment of area about a line through the centroid of some common shapes.

Shape	Area A	2nd moment of area, I_{GG}, about an axis through the centroid
Rectangle 	bd	$\frac{bd^3}{12}$
Triangle 	$\frac{bd}{2}$	$\frac{bd^3}{36}$
Circle 	πR^2	$\frac{\pi R^4}{4}$
Semicircle 	$\frac{\pi R^2}{2}$	$0.1102R^4$

Lateral position of Centre of Pressure

If the shape is symmetrical the centre of pressure lies on the line of symmetry. But if it is not symmetrical its position must be found by taking moments about the line OG in the same way as we took moments along the line through O, i.e.

$$R \times d = \text{Sum of th e moments of the for ce on all elemnts of } \delta A \text{ about OG}$$
$$= \sum \rho g z \delta A x$$

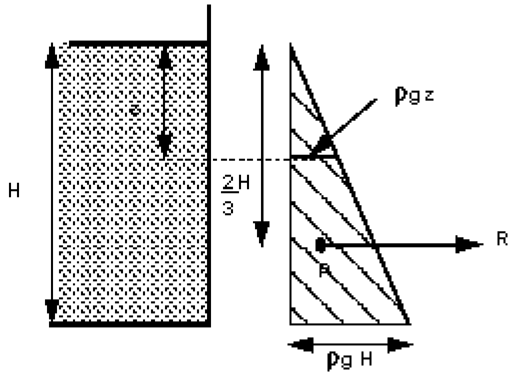
but we have $R = \rho g A \bar{z}$ so

$$d = \frac{\sum \delta A z x}{A\bar{z}}$$

3. Submerged vertical surface - Pressure diagrams

For vertical walls of constant width it is usually much easier to find the resultant force and centre of pressure. This is done graphically by means of a pressure diagram.

Consider the tank in the diagram below having vertical walls and holding a liquid of density ρ to a depth of *H*. To the right can be seen a graphical representation of the (gauge) pressure change with depth on one of the vertical walls. Pressure increases from zero at the surface linearly by $p = \rho g z$, to a maximum at the base of $p = \rho g H$.



Pressure diagram for vertical wall.

The area of this triangle represents the **resultant force per unit width** on the vertical wall, using SI units this would have units of Newtons per metre. So

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times AB \times BC \\ &= \frac{1}{2} H \rho g H \\ &= \frac{1}{2} \rho g H^2 \end{aligned}$$

Resultant force per unit width

$$R = \frac{1}{2} \rho g H^2 \qquad (N/m)$$

The force acts through the centroid of the pressure diagram. For a triangle the centroid is at 2/3 its height, i.e. in the figure above the resultant force acts horizontally through the point $z = \frac{2}{3} H$.

For a vertical plane the depth to the centre of pressure is given by

$$D = \frac{2}{3}H$$

This can be checked against the previous method:

The resultant force is given by:

$$\begin{aligned} R &= \rho g A\bar{z} = \rho g A\bar{x}\sin\theta \\ &= \rho g\,(H \times 1)\frac{H}{2}\sin\theta \\ &= \frac{1}{2}\rho g H^2 \end{aligned}$$

and the depth to the centre of pressure by:

$$D = \sin\theta\left(\frac{I_o}{A\bar{x}}\right)$$

and by the parallel axis theorem (with width of 1)

$$\begin{aligned} I_o &= I_{GG} + A\bar{x}^2 \\ &= \frac{1 \times H^3}{12} + 1 \times H\left(\frac{H}{2}\right)^2 \\ &= \frac{H^3}{12} + \frac{H^3}{4} \\ &= \frac{H^3}{3} \end{aligned}$$

Giving depth to the centre of pressure

$$\begin{aligned} D &= \left(\frac{H^3/3}{H^2/2}\right) \\ &= \frac{2}{3}H \end{aligned}$$

These two results are identical to the pressure diagram method.

The same pressure diagram technique can be used when combinations of liquids are held in tanks (e.g. oil floating on water) with position of action found by taking moments of the individual resultant forces for each fluid. Look at the examples to examine this area further.

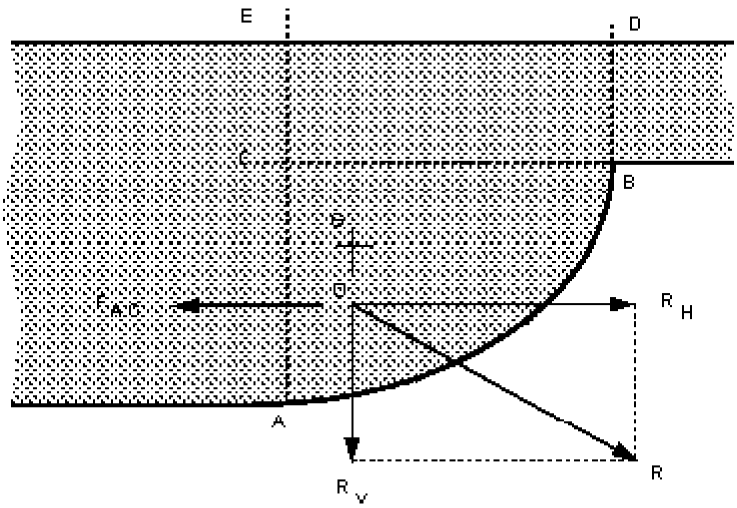
More complex pressure diagrams can be draw for non-rectangular or non-vertical planes but it is usually far easier to use the moments method.

4. Resultant force on a submerged curved surface

As stated above, if the surface is curved the forces on each element of the surface will not be parallel and must be combined using some vectorial method.

It is most straightforward to calculate the horizontal and vertical components and combine these to obtain the resultant force and its direction. (This can also be done for all three dimensions, but here we will only look at one vertical plane).

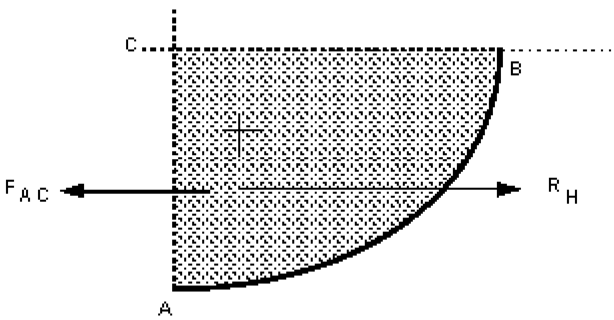
In the diagram below the liquid is resting on top of a curved base.



The element of fluid ABC is equilibrium (as the fluid is at rest).

<u>Horizontal forces</u>

Considering the horizontal forces, none can act on CB as there are no shear forces in a static fluid so the forces would act on the faces AC and AB as shown below.



We can see that the horizontal force on AC, F_{AC}, must equal and be in the opposite direction to the resultant force R_H on the curved surface.

As AC is the projection of the curved surface AB onto a vertical plane, we can generalise this to say

---

The resultant horizontal force of a fluid above a curved surface is:

R_H = Resultant force on the projection of the curved surface onto a vertical plane.

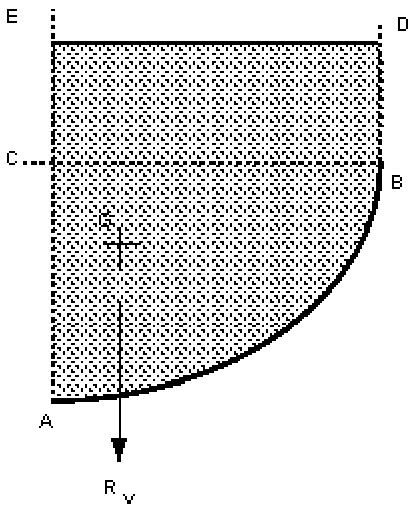
We know that the force on a vertical plane must act horizontally (as it acts normal to the plane) and that R_H must act through the same point. So we can say

R_H acts horizontally through the centre of pressure of the projection of the curved surface onto an vertical plane.

Thus we can use the pressure diagram method to calculate the position and magnitude of the resultant horizontal force on a two dimensional curved surface.

Vertical forces

The diagram below shows the vertical forces which act on the element of fluid above the curved surface.



There are no shear force on the vertical edges, so the vertical component can only be due to the weight of the fluid. So we can say

The resultant vertical force of a fluid above a curved surface is:

R_V = Weight of fluid directly above the curved surface.

and it will act vertically downward through the centre of gravity of the mass of fluid.

Resultant force

The overall resultant force is found by combining the vertical and horizontal components vectorialy,

Resultant force

$$R = \sqrt{R_H^2 + R_V^2}$$

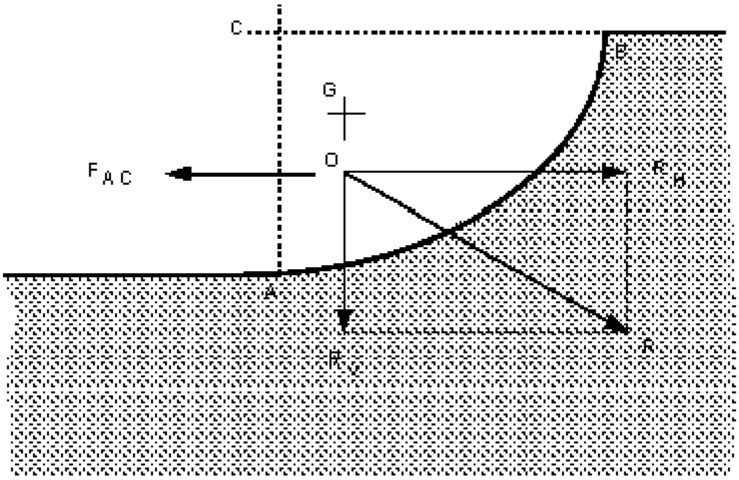
And acts through O at an angle of θ.

The angle the resultant force makes to the horizontal is

$$\theta = \tan^{-1}\left(\frac{R_V}{R_H}\right)$$

The position of O is the point of integration of the horizontal line of action of R_H and the vertical line of action of R_V.

*What are the forces if the fluid is **below** the curved surface*? This situation may occur or a curved sluice gate for example. The figure below shows a situation where there is a curved surface which is experiencing fluid pressure from below.



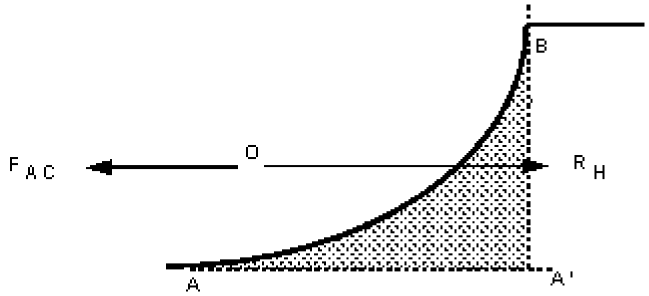
The calculation of the forces acting from the fluid below is very similar to when the fluid is above.

<u>Horizontal force</u>

From the figure below we can see the only two horizontal forces on the area of fluid, which is in equilibrium, are the horizontal reaction force which is equal and in the opposite direction to the pressure force on the vertical plane A'B. The resultant horizontal force, R_H acts as shown in the diagram. Thus we can say:

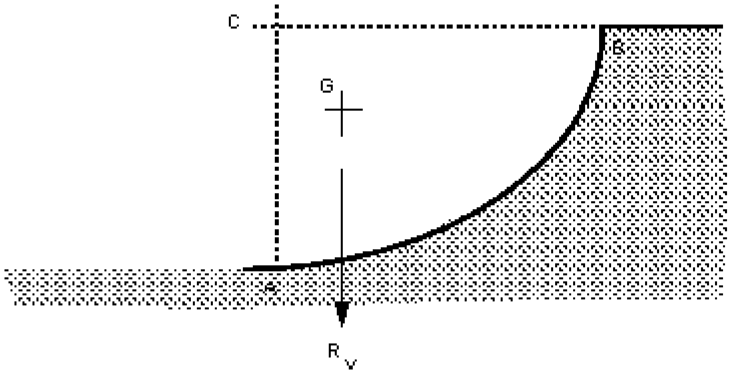
The resultant horizontal force of a fluid below a curved surface is:

$$R_H = \text{Resultant force on the projection of the curved surface on a vertical plane}$$



Vertical force

The vertical force are acting are as shown on the figure below. If the curved surface were removed and the area it were replaced by the fluid, the whole system would be in equilibrium. Thus the force required by the curved surface to maintain equilibrium is equal to that force which the fluid above the surface would exert - i.e. the weight of the fluid.



Thus we can say:

The resultant vertical force of a fluid below a curved surface is:

R_v =Weight of the *imaginary* volume of fluid vertically above the curved surface.

The resultant force and direction of application are calculated in the same way as for fluids above the surface:

Resultant force

$$R = \sqrt{R_H^2 + R_V^2}$$

And acts through O at an angle of θ.

The angle the resultant force makes to the horizontal is

$$\theta = \tan^{-1}\left(\frac{R_V}{R_H}\right)$$

1. Example of the pressure and head relationship:

What is a pressure of 500 *kNm*$^{-2}$

A) In head of water of density, ρ = 1000 kgm^{-3}

Use *p* = *ρgh*,

$$h = \frac{p}{\rho g} = \frac{500 \times 10^3}{1000 \times 9.81} = 50.95m \text{ of water}$$

B) In head of Mercury density ρ = 13.6 x 10^3 kgm^{-3}.

$$h = \frac{500 \times 10^3}{13.6 \times 10^3 \times 9.81} = 3.75m \text{ of Mercury}$$

C) In head of a fluid with relative density γ = 8.7.

(remember $\rho = \gamma \times \rho_{water}$)

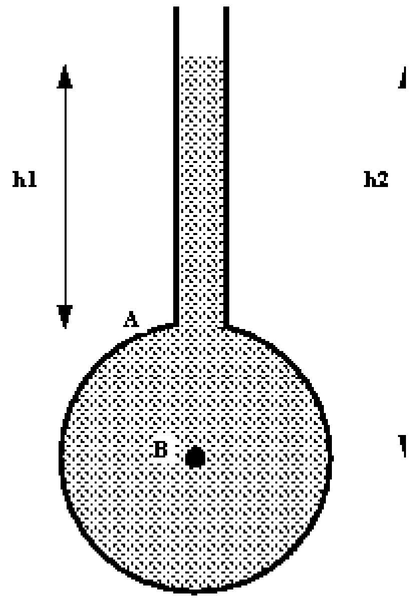
$$h = \frac{500 \times 10^3}{(8.7 \times 1000) \times 9.81} = 5.86m \text{ of fluid } \gamma = 8.7$$

---

2. Example of a Piezometer.

What is the maximum gauge pressure of water that can be measured by a Piezometer of height 1.5m?

And if the liquid had a relative density of 8.5 what would the maximum measurable gauge pressure?



gauge pressure $p = \rho g h$

$\rho = \rho_{water} \times$ *relative density*

The maximum measurable pressure is when the tube is completely full (*h=1.5m*).

Any higher and the tube will overflow.

$p = (0.85 \times 10^3) \times 9.81 \times 1.5$

$p = 12\ 508\ N/m^2$ ***(or Pa)***

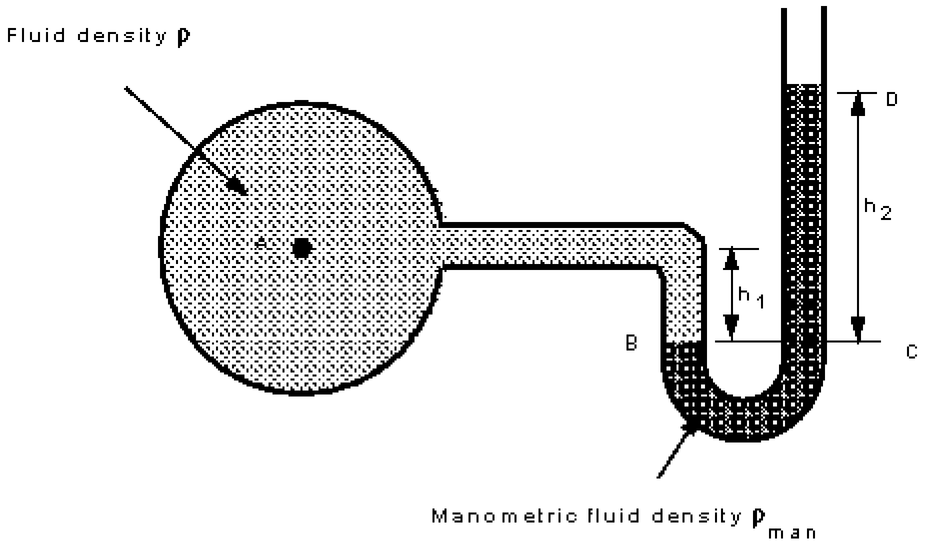
$p = 12.5\ kN/m^2$ ***(or kPa)***

3. An example of the U-Tube manometer.

Using a u-tube manometer to measure gauge pressure of fluid density $\rho = 700\ kg/m^3$, and the manometric fluid is mercury, with a relative density of 13.6.

What is the gauge pressure if:

1. **$h_1 = 0.4m$ and $h_2 = 0.9m$?**
1. **h_1 stayed the same but $h_2 = -0.1m$?**



$p_B = p_C$

$p_B = p_A + \rho g h_1$

$p_B = p_{Atmospheric} + \rho_{man} g h_2$

We are measuring *gauge* pressure so $p_{atmospheric} = 0$

$p_A = \rho_{man} g h_2 - \rho g h_1$

a) p_A = 13.6* x *10^3* x *9.81* x *0.9 - 700* x *9.81* x *0.4

= 117 327 N, 117.3 kN (1.17 bar)

b) p_A = 13.6* x *10^3* x *9.81* x *(-0.1) - 700* x *9.81* x *0.4

= -16 088.4 N, -16 kN (-0.16 bar)

The negative sign indicates that the pressure is

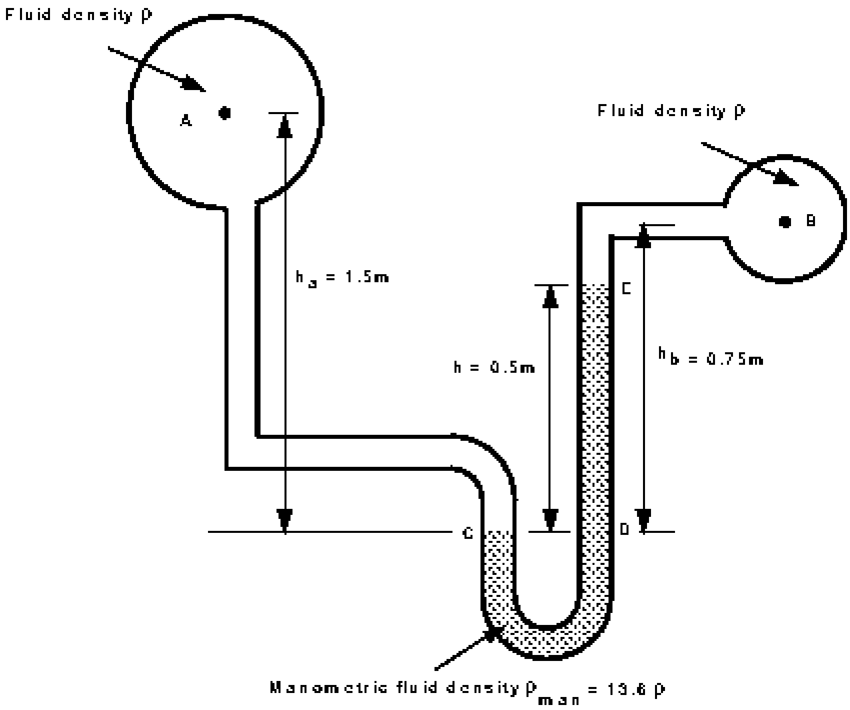
below atmospheric

---

4. Example of the u-tube for pressure difference measurement

In the figure below two pipes containing the same fluid of density $\rho = 990\ kg/m^3$ are connected using a u-tube manometer.

What is the pressure between the two pipes if the manometer contains fluid of relative density 13.6?



$$p_C = p_D$$

$$p_C = p_A + \rho g\, h_A$$

$$p_D = p_B + \rho g\,(h_B - h) + \rho_{man}\, g\, h$$

$$p_A - p_B = \rho g\,(h_B - h_A) + hg(\rho_{man} - \rho)$$

$$= 990 \times 9.81 \times (0.75-1.5) + 0.5 \times 9.81 \times (13.6-0.99) \times 10^3$$

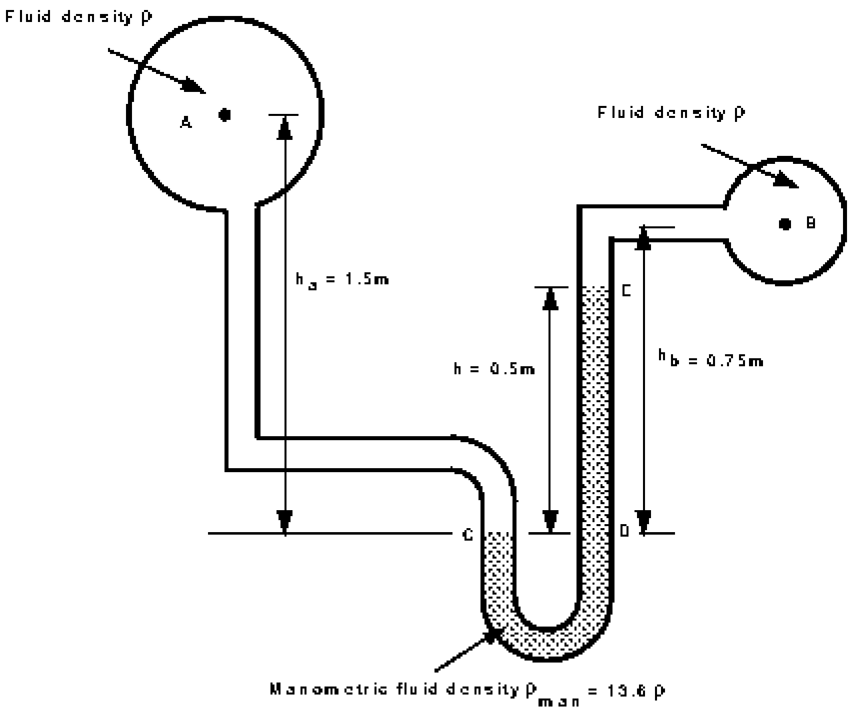
$$= -7284 + 61852$$

$$= 54\,568\ \text{N/m}^2\ \text{(or Pa or 0.55 bar)}$$

4. Example of the u-tube for pressure difference measurement

In the figure below two pipes containing the same fluid of density ρ = 990 kg/m^3 are connected using a u-tube manometer.

What is the pressure between the two pipes if the manometer contains fluid of relative density 13.6?



$p_C = p_D$

$p_C = p_A + \rho g\, h_A$

$p_D = p_B + \rho g\, (h_B - h) + \rho_{man}\, g\, h$

$p_A - p_B = \rho g\, (h_B - h_A) + hg(\rho_{man} - \rho)$

$= 990 \times 9.81 \times (0.75-1.5) + 0.5 \times 9.81 \times (13.6-0.99) \times 10^3$

$= -7284 + 61852$

= 54 568 N/m² (or Pa or 0.55 bar)

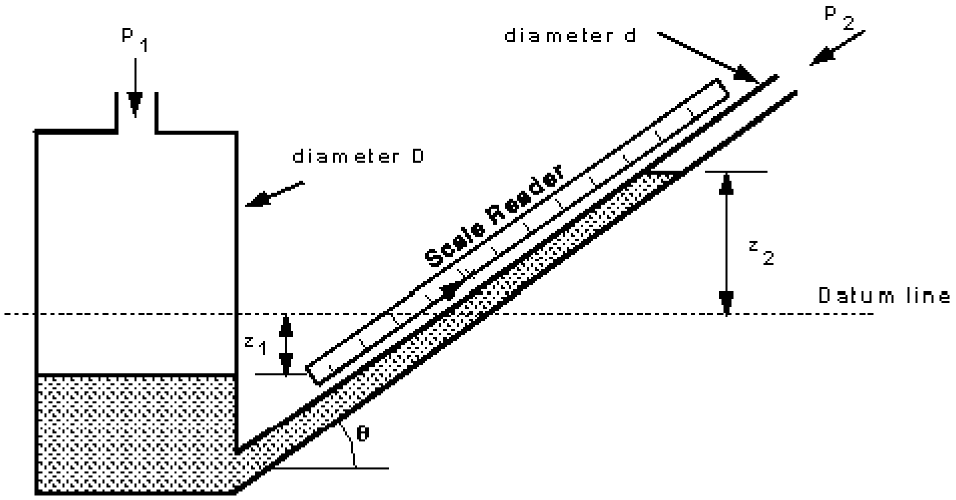
Go back to the main index page

Example of an inclined manometer.

An inclined tube manometer consists of a vertical cylinder 35mm diameter. At the bottom of this is connected a tube 5mm in diameter inclined upward at an angle of 15 to the horizontal, the top of this tube is connected to an air duct. The vertical cylinder is open to the air and the manometric fluid has relative density 0.785.

Determine the pressure in the air duct if the manometric fluid moved 50mm along the inclined tube.

What is the error if the movement of the fluid in the vertical cylinder is ignored?



Use this equation derived in the lectures

$$p_1 - p_2 = \rho g h = \rho g(z_1 + z_2)$$

for a manometer where $\rho_{man} >> \rho$.

where

$$z_2 = x \sin \theta,$$

and

$$A_1 z_1 = a_2 x$$
$$z_1 = x (d/D)^2$$

where x is the reading on the manometer scale.

p_1 is atmospheric i.e. $p_1 = 0$

$$p_2 = -\rho g x\left(\sin\theta + \left(\frac{d}{D}\right)^2\right)$$

And $x = -50mm = -0.05m$,

$$-p_2 = 0.785 \times 10^3 \times 9.81 \times (-0.05)\left[\sin 15 + \left(\frac{0.005}{0.035}\right)^2\right]$$

$$p_2 = 107.2 \;\; N$$

If the movement in the large cylinder is ignored the term $(d/D)^2$ will disappear:

$$p_1 - p_2 = \rho g x \sin\theta$$
$$p_2 = 0.785 \times 10^3 \times 9.81 \times 0.05 \times \sin 15$$
$$= 99.66 \;\; N$$

So the error induced by this assumption is

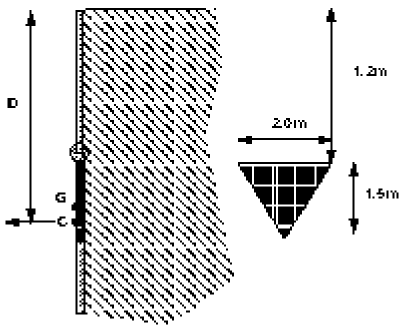
$$error = \frac{107.2 - 99.66}{102.2}100 = 7.3\%$$

Go back to the main index page

6. An example of force on submerged plane.

A tank holding water has a triangular gate, hinged at the top, in one wall.

Find the moment at the hinge required to keep this triangular gate closed.

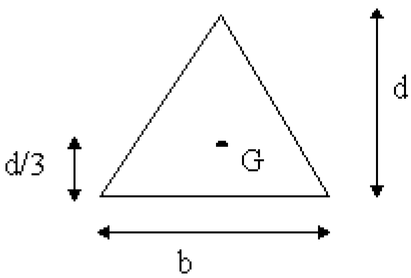


Resultant force

R = pressure at centroid x ***Area***

$= \rho g \bar{z} \times A$

Where $\bar{z}$ is the depth to the centroid (point G above).



$$R = \rho g\left(1.2+\frac{d}{3}\right)\left(\frac{bd}{2}\right)$$

$$= 1000\times 9.81\left(1.2+\frac{1.5}{3}\right)\left(\frac{2.0\times 1.5}{2}\right)$$

$$= 25015.5\ N$$

This force acts through the <u>centre of pressure</u>.

From the lecture notes, depth to centre of pressure

$$D = \sin\theta \frac{I_{oo}}{A\overline{x}}$$

As the wall is vertical *sin* θ *= 1* and $\overline{x} = \overline{z}$

The parallel axis theorem says, $I_{oo} = I_{GG} + A\overline{x}^2$

$$D = \frac{I_{GG}}{A\overline{x}} + \overline{x}$$

And for a triangle, $I_{GG} = \dfrac{bd^3}{36}$

$$D = \frac{2.0\times 1.5^3}{36}\frac{2.0}{1.5\times 2.0}\frac{1}{1.2+1.5/3} + (1.2+1.5/3)$$

$$= 1.77m$$

The moment on the hinge from the water is

$$
\begin{aligned}
moment &= R \times (D - 1.2) \\
&= 25015.5 \times 0.57 \\
&= 14259\, Nm
\end{aligned}
$$

Which is clockwise.

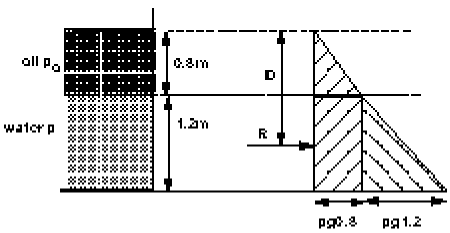
The moment required to keep the gate closed is equal but anticlockwise.

Go back to the main index page

---

7. Example of pressure diagram method for pressure on a vertical plane.

Find the position and magnitude of the resultant force on this vertical wall of a tank which has oil, of relative density 0.8, floating on water as shown.



Draw the pressure diagram as shown to the right.

The resultant force (per unit length) is simply the area of the pressure diagram.

It is convenient to split the area into three, and sum the answers.

$F_1 = A_1 = (0.8 \times 10^3) \times 9.81 \times 0.8 \times 0.8 \times 0.5 = 2511.36$

$F_2 = A_2 = (0.8 \times 10^3) \times 9.81 \times 0.8 \times 1.2 = 7534.08$

$F_3 = A_3 = (10^3) \times 9.81 \times 1.2 \times 1.2 \times 0.5 = 7063.20$

$R = F_1 + F_2 + F_3 = 17108.64\ N/m$

R **acts horizontally through the centroid of the pressure diagram.**

This position can be found by taking moments of the individual forces and equating the sum of these to the moment caused by the resultant force.

$RD = F_1d_1 + F_2d_2 + F_3d_3$

The individual forces act horizontally through the centroid of the individual areas.

$d_1 = 0.8 \times 2/3 = 0.533$

$d_2 = 0.8 + 1.2/2 = 1.4$

$d_3 = 0.8 +1.2 \times 2/3 = 1.6$

$17108D = 2511 \times 0.53 + 7534 \times 1.4 + 7063 \times 1.6$

$= 1339 + 1054+11301$

$= 23188$

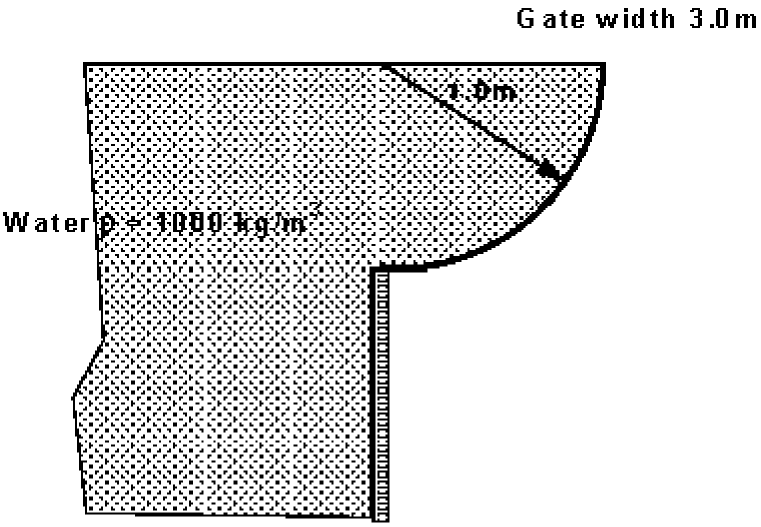
$D = 1.36\ m$

Go back to the main index page

---

8. An example of force on a curved wall

Find the magnitude and direction of the resultant force of water on a quadrant gate as shown below.



Horizontal force, ***R_H = Force on projection of curved surface on to a vertical plane***

$= 0.5 \times \rho g h^2 \times width$

$= 0.5 \times 1000 \times 9.81 \times 1^2 \times 3$

$= 14715\ N$

Vertical force, R_V = weight of fluid above surface

$= \rho g \times Volume$

$= 1000 \times 9.81 \times (\pi r^2/4) \times 3$

$= 23114\ N$

Resultant force $R = \sqrt{R_V^2 + R_H^2} = 27400\ N$

At an angle

$$\theta = \tan^{-1}\left(\frac{R_V}{R_H}\right) = \frac{23144}{14715} = 57^\circ 31' = 57.51^\circ$$

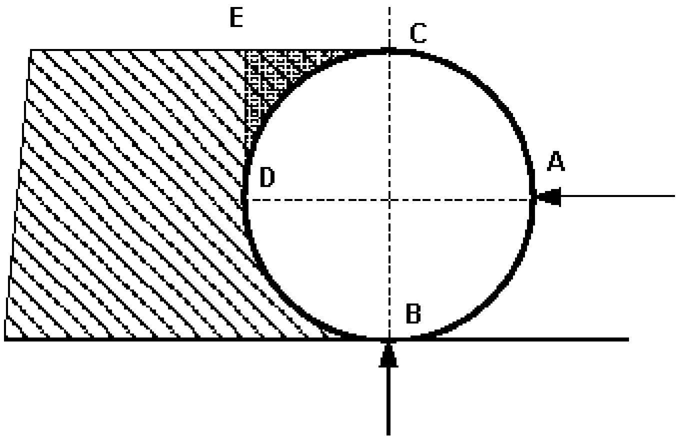
Go back to the main index page

---

9. An example of force on a curved sluice gate with fluid both above and below the gate.

A 1.5m long cylinder, radius 1m, lies as shown in the figure. It holds back oil of relative density 0.8. If the cylinder has a mass of 2250 kg find

a) the reaction at A b) the reaction at B



Horizontal force R_H = projection of vertical plane

Reaction at A = $-R_H = 0.5 \times \rho g H^2 \times Length$

$= 0.5 \times 0.8 \times 10^3 \times 9.81 \times 2^2 \times 1.5$

= 23544 N, to the left

Vertical force = Reaction at B

R_V = Force due to weight of fluid, DCE, above (down)

+ Force due to fluid below BD (upward)

+ Force due to weight of cylinder

Force of fluid above = area of sector DCE x length x ρg

Force from below = area of real or imaginary fluid above BD

= area of BDEC x length x ρg

Taking downward as positive

$R_V = 0.8 \times 10^3 \times 9.81 \times (11 - \pi r^2/4) \times 1.5$

$- 0.8 \times 10^3 \times 9.81 \times (11 + \pi r^2/4)) \times 1.5$

$+ 22509.81$

$= 2526 -$

$= 3580\ N$

Reaction at $B = 3580\ N$**, *vertically up***

The resultant and angle of application are given by:

$$R = \sqrt{R_H^2 + R_V^2} = 23815\ N$$

$$\theta = \tan^{-1}\left(\frac{3580}{23544}\right)$$

$$= 8.64^\circ$$

$$= 8^\circ\ 38'$$

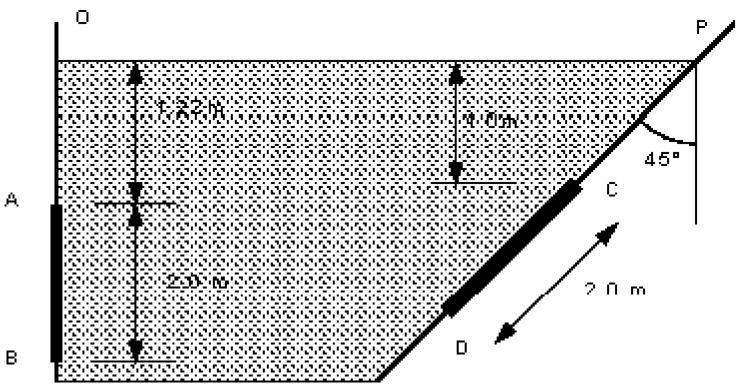
Go back to the main index page

Statics Examples

Pressure and Manometers

1. What will be the (a) the gauge pressure and (b) the absolute pressure of water at depth 12m below the surface? ρ_{water} = 1000 kg/m^3, and p $_{atmosphere}$ = 101kN/m^2.
   [117.72 kN/m^2, 218.72 kN/m^2]
2. At what depth below the surface of oil, relative density 0.8, will produce a pressure of 120 kN/m^2? What depth of water is this equivalent to?
   [15.3m, 12.2m]

3. What would the pressure in kN/m^2 be if the equivalent head is measured as 400mm of (a) mercury γ=13.6 (b) water (c) oil specific weight 7.9 kN/m^3 (d) a liquid of density 520 kg/m^3?
   [53.4 kN/m^2, 3.92 kN/m^2, 3.16 kN/m^2, 2.04 kN/m^2]
4. A manometer connected to a pipe indicates a negative gauge pressure of 50mm of mercury. What is the absolute pressure in the pipe in Newtons per square metre if the atmospheric pressure is 1 bar?
   [93.3 kN/m^2]
5. What height would a water barometer need to be to measure atmospheric pressure?
   [>10m]
6. An inclined manometer is required to measure an air pressure of 3mm of water to an accuracy of +/- 3%. The inclined arm is 8mm in diameter and the larger arm has a diameter of 24mm. The manometric fluid has density 740 kg/m^3 and the scale may be read to +/- 0.5mm.
   What is the angle required to ensure the desired accuracy may be achieved?
   [12 39']
7. Determine the resultant force due to the water acting on the 1m by 2m rectangular area AB shown in the diagram below.
   (On the diagram distance OA is 1.22m and AB is 2.0m)
   [43 560 N, 2.37m from O



8. Determine the resultant force due to the water acting on the 1.25m by 2.0m triangular area CD shown in the figure above. The apex of the triangle is at C.
   (On the diagram depth to point C is 1.0m and the distance CD is 2.0m)
   [23.81$0^3$N, 2.821m from P]

Forces on submerged surfaces

1. Obtain an expression for the depth of the centre of pressure of a plane surface wholly submerged in a fluid and inclined at an angle to the free surface of the liquid.
   A horizontal circular pipe, 1.25m diameter, is closed by a butterfly disk which rotates about a horizontal axis through its centre. Determine the torque which would have to be applied to the disk spindle to keep the disk closed in a vertical position when there is a 3m head of fresh water above the axis.
   [1176 Nm]
2. A dock gate is to be reinforced with three horizontal beams. If the water acts on one side only, to a depth of 6m, find the positions of the beams measured

from the water surface so that each will carry an equal load. Give the load per meter.
[58 860 N/m, 2.31m, 4.22m, 5.47m]

3. The profile of a masonry dam is an arc of a circle, the arc having a radius of 30m and subtending an angle of 60 at the centre of curvature which lies in the water surface. Determine (a) the load on the dam in N/m length, (b) the position of the line of action to this pressure.
[4.28 10^6 N/m length at depth 19.0m]
4. The arch of a bridge over a stream is in the form of a semi-circle of radius 2m. the bridge width is 4m. Due to a flood the water level is now 1.25m above the crest of the arch. Calculate (a) the upward force on the underside of the arch, (b) the horizontal thrust on one half of the arch.
[263.6 kN, 176.6 kN]
5. The face of a dam is vertical to a depth of 7.5m below the water surface then slopes at 30 to the vertical. If the depth of water is 17m what is the resultant force per metre acting on the whole face?
[1563.29 kN]
6. A tank with vertical sides is square in plan with 3m long sides. The tank contains oil of relative density 0.9 to a depth of 2.0m which is floating on water a depth of 1.5m. Calculate the force on the walls and the height of the centre of pressure from the bottom of the tank.
[165.54 kN, 1.15m]

1. Fluid Dynamics

Objectives

- Introduce concepts necessary to analyses fluids in motion
- Identify differences between Steady/unsteady uniform/non-uniform compressible/incompressible flow
- Demonstrate streamlines and stream tubes
- Introduce the Continuity principle through conservation of mass and control volumes
- Derive the Bernoulli (energy) equation
- Demonstrate practical uses of the Bernoulli and continuity equation in the analysis of flow
- Introduce the momentum equation for a fluid
- Demonstrate how the momentum equation and principle of conservation of momentum is used to predict forces induced by flowing fluids

This section discusses the analysis of fluid in motion - fluid dynamics. The motion of fluids can be predicted in the same way as the motion of solids are predicted using the fundamental laws of physics together with the physical properties of the fluid.

It is not difficult to envisage a very complex fluid flow. Spray behind a car; waves on beaches; hurricanes and tornadoes or any other atmospheric phenomenon are all example of highly complex fluid flows which can be analysed with varying degrees of success (in some cases hardly at all!). There are many common situations which are easily analysed.

2. Uniform Flow, Steady Flow

It is possible - and useful - to classify the type of flow which is being examined into small number of groups.

If we look at a fluid flowing under normal circumstances - a river for example - the conditions at one point will vary from those at another point (e.g. different velocity) we have non-uniform flow. If the conditions at one point vary as time passes then we have unsteady flow.

Under some circumstances the flow will not be as changeable as this. He following terms describe the states which are used to classify fluid flow:

- *uniform flow:* If the flow velocity is the same magnitude and direction at every point in the fluid it is said to be *uniform*.
- *non-uniform:* If at a given instant, the velocity is **not** the same at every point the flow is *non-uniform*. (In practice, by this definition, every fluid that flows near a solid boundary will be non-uniform - as the fluid at the boundary must take the speed of the boundary, usually zero. However if the size and shape of the of the cross-section of the stream of fluid is constant the flow is considered *uniform*.)
- *steady:* A steady flow is one in which the conditions (velocity, pressure and cross-section) may differ from point to point but DO NOT change with time.
- *unsteady:* If at any point in the fluid, the conditions change with time, the flow is described as *unsteady*. (In practise there is always slight variations in velocity and pressure, but if the average values are constant, the flow is considered *steady*.

Combining the above we can classify any flow in to one of four type:

1. *Steady uniform flow*. Conditions do not change with position in the stream or with time. An example is the flow of water in a pipe of constant diameter at constant velocity.
2. *Steady non-uniform flow*. Conditions change from point to point in the stream but do not change with time. An example is flow in a tapering pipe with constant velocity at the inlet - velocity will change as you move along the length of the pipe toward the exit.
3. *Unsteady uniform flow*. At a given instant in time the conditions at every point are the same, but will change with time. An example is a pipe of constant diameter connected to a pump pumping at a constant rate which is then switched off.
4. *Unsteady non-uniform flow*. Every condition of the flow may change from point to point and with time at every point. For example waves in a channel.

If you imaging the flow in each of the above classes you may imagine that one class is more complex than another. And this is the case - *steady uniform flow* is by far the most simple of the four. You will then be pleased to hear that this course is restricted to only this class of flow. We will not be encountering any non-uniform or unsteady

effects in any of the examples (except for one or two quasi-time dependent problems which can be treated at steady).

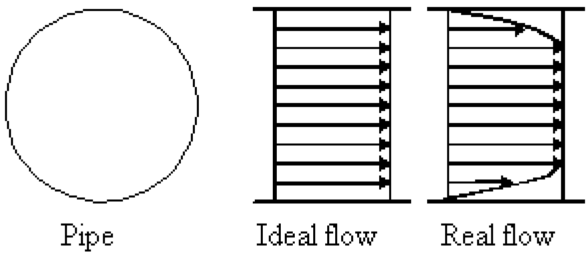
3. Compressible or Incompressible

All fluids are compressible - even water - their density will change as pressure changes. Under steady conditions, and provided that the changes in pressure are small, it is usually possible to simplify analysis of the flow by assuming it is incompressible and has constant density. As you will appreciate, liquids are quite difficult to compress - so under most steady conditions they are treated as incompressible. In some unsteady conditions very high pressure differences can occur and it is necessary to take these into account - even for liquids. Gasses, on the contrary, are very easily compressed, it is essential in most cases to treat these as compressible, taking changes in pressure into account.

4. Three-dimensional flow

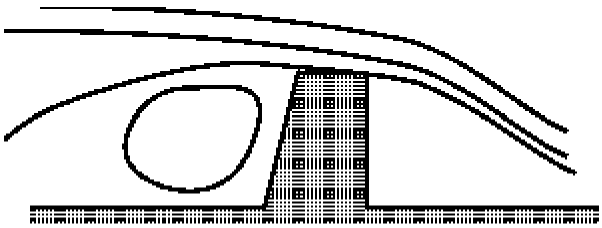
Although in general all fluids flow three-dimensionally, with pressures and velocities and other flow properties varying in all directions, in many cases the greatest changes only occur in two directions or even only in one. In these cases changes in the other direction can be effectively ignored making analysis much more simple.

Flow is *one dimensional* if the flow parameters (such as velocity, pressure, depth etc.) at a given instant in time only vary in the direction of flow and not across the cross-section. The flow may be unsteady, in this case the parameter vary in time but still not across the cross-section. An example of one-dimensional flow is the flow in a pipe. Note that since flow must be zero at the pipe wall - yet non-zero in the centre - there is a difference of parameters across the cross-section. Should this be treated as two-dimensional flow? Possibly - but it is only necessary if very high accuracy is required. A correction factor is then usually applied.



One dimensional flow in a pipe.

Flow is *two-dimensional* if it can be assumed that the flow parameters vary in the direction of flow and in one direction at right angles to this direction. Streamlines in two-dimensional flow are curved lines on a plane and are the same on all parallel planes. An example is flow over a weir foe which typical streamlines can be seen in the figure below. Over the majority of the length of the weir the flow is the same - only at the two ends does it change slightly. Here correction factors may be applied.

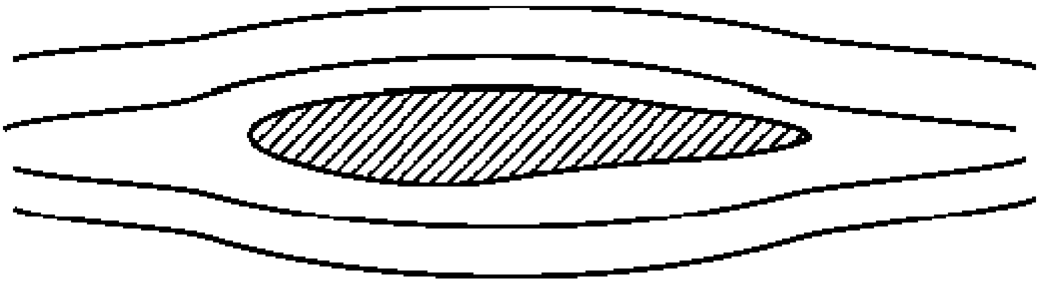


Two-dimensional flow over a weir.

In this course we will **only** be considering steady, incompressible one and two-dimensional flow.

5. Streamlines and streamtubes

In analysing fluid flow it is useful to visualise the flow pattern. This can be done by drawing lines joining points of equal velocity - velocity contours. These lines are know as *streamlines*. Here is a simple example of the streamlines around a cross-section of an aircraft wing shaped body:



Streamlines around a wing shaped body

When fluid is flowing past a solid boundary, e.g. the surface of an aerofoil or the wall of a pipe, fluid obviously does not flow into or out of the surface. So very close to a boundary wall the flow direction must be parallel to the boundary.

- *Close to a solid boundary streamlines are parallel to that boundary*

At all points the direction of the streamline is the direction of the fluid velocity: this is how they are defined. Close to the wall the velocity is parallel to the wall so the streamline is also parallel to the wall.

It is also important to recognise that the position of streamlines can change with time - this is the case in unsteady flow. In steady flow, the position of streamlines does not change.

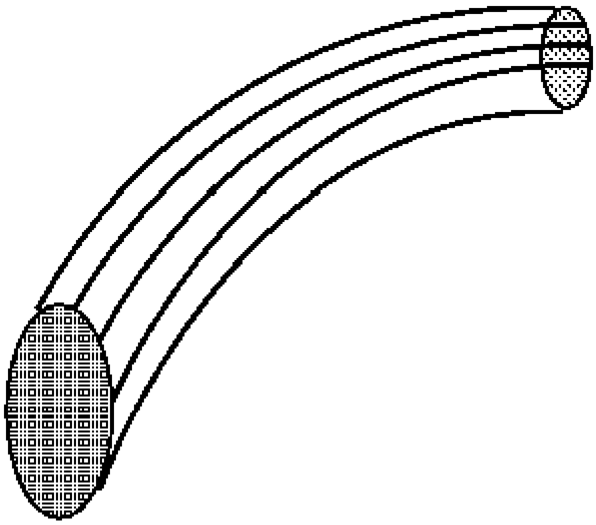
Some things to know about streamlines

- Because the fluid is moving in the same direction as the streamlines, fluid can not cross a streamline.

- Streamlines can not cross each other. If they were to cross this would indicate two different velocities at the same point. This is not physically possible.

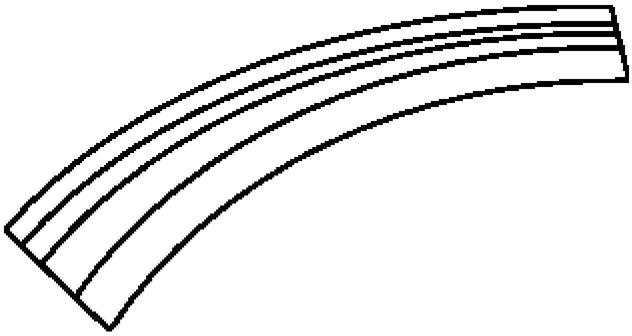
- The above point implies that any particles of fluid starting on one streamline will stay on that same streamline throughout the fluid.

A useful technique in fluid flow analysis is to consider only a part of the total fluid in isolation from the rest. This can be done by imagining a tubular surface formed by streamlines along which the fluid flows. This tubular surface is known as a *streamtube*.



A Streamtube

And in a two-dimensional flow we have a streamtube which is flat (in the plane of the paper):



A two dimensional version of the streamtube

The "walls" of a streamtube are made of streamlines. As we have seen above, fluid cannot flow across a streamline, so fluid cannot cross a streamtube wall. The streamtube can often be viewed as a solid walled pipe. A streamtube is **not** a pipe - it differs in unsteady flow as the walls will move with time. And it differs because the "wall" is moving with the fluid

---

Go back to the main index page

Go back to the main index page

Continuity and Conservation of Matter

1. Mass flow rate

If we want to measure the rate at which water is flowing along a pipe. A very simple way of doing this is to catch all the water coming out of the pipe in a bucket over a fixed time period. Measuring the weight of the water in the bucket and dividing this by the time taken to collect this water gives a rate of accumulation of mass. This is know as the *mass flow rate.*

For example an empty bucket weighs 2.0kg. After 7 seconds of collecting water the bucket weighs 8.0kg, then:

$$\begin{aligned} \text{mass flow rate} = \dot{m} &= \frac{\text{mass of fluid in bucket}}{\text{time taken to collect the fluid}} \\ &= \frac{8.0 - 2.0}{7} \\ &= 0.857 kg/s \quad (kg s^{-1}) \end{aligned}$$

Performing a similar calculation, if we know the mass flow is 1.7kg/s, how long will it take to fill a container with 8kg of fluid?

$$\begin{aligned} \text{time} &= \frac{\text{mass}}{\text{mass flow rate}} \\ &= \frac{8}{1.7} \\ &= 4.7 s \end{aligned}$$

2. Volume flow rate - Discharge.

More commonly we need to know the volume flow rate - this is more commonly know as *discharge.* (It is also commonly, but inaccurately, simply called flow rate). The symbol normally used for discharge is Q. The discharge is the volume of fluid flowing per unit time. Multiplying this by the density of the fluid gives us the mass flow rate. Consequently, if the density of the fluid in the above example is 850 $kg m^3$ then:

$$\begin{aligned} \text{discharge,} \quad Q &= \frac{\text{volume of fluid}}{\text{time}} \\ &= \frac{\text{mass of fluid}}{\text{density} \times \text{time}} \\ &= \frac{\text{mass flow rate}}{\text{density}} \\ &= \frac{0.857}{850} \\ &= 0.001008\, m^3/s \quad (m^3 s^{-1}) \\ &= 1.008 \times 10^{-3} m^3/s \\ &= 1.008\, l/s \end{aligned}$$

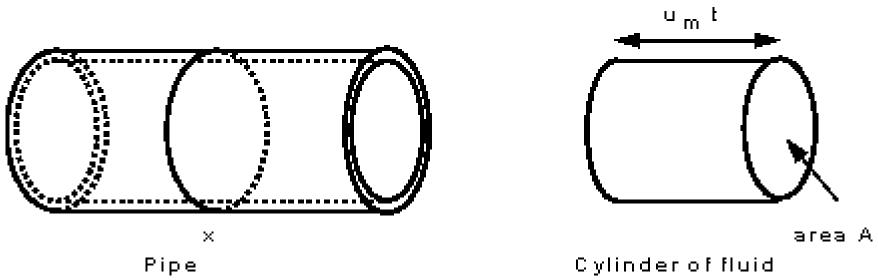
An important aside about units should be made here:

As has already been stressed, we must always use a consistent set of units when applying values to equations. It would make sense therefore to always quote the values in this consistent set. This set of units will be the SI units. Unfortunately, and this is the case above, these actual practical values are very small or very large (0.001008m^3/s is very small). These numbers are difficult to imagine physically. In these cases it is useful to use *derived units*, and in the case above the useful derived unit is the litre.

(*1 litre = 1.0* $10^{-3} m^3$). So the solution becomes $1.008\, l/s$. It is far easier to imagine 1 litre than *1.0* $10^{-3} m^3$. Units must always be checked, and converted if necessary to a consistent set before using in an equation.

3. Discharge and mean velocity

If we know the size of a pipe, and we know the discharge, we can deduce the mean velocity



Discharge in a pipe

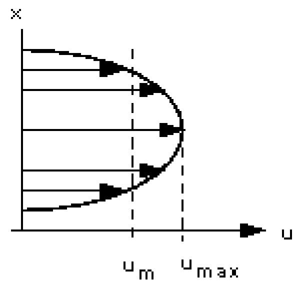
If the area of cross section of the pipe at point X is A, and the mean velocity here is u_m. During a time t, a cylinder of fluid will pass point X with a volume Au_mt. The volume per unit time (the discharge) will thus be

$$Q = \frac{\text{volume}}{\text{time}} = \frac{A \times u_m \times t}{t}$$
$$Q = Au_m$$

So if the cross-section area, A, is $1.2 \times 10^{-3} m^2$ and the discharge, Q is $24 l/s$, then the mean velocity, u_m, of the fluid is

$$\begin{aligned} u_m &= \frac{Q}{A} \\ &= \frac{2.4 \times 10^{-3}}{1.2 \times 10^{-3}} \\ &= 2.0 m/s \end{aligned}$$

Note how carefully we have called this the *mean* velocity. This is because the velocity in the pipe is not constant across the cross section. Crossing the centreline of the pipe, the velocity is zero at the walls increasing to a maximum at the centre then decreasing symmetrically to the other wall. This variation across the section is known as the velocity profile or distribution. A typical one is shown in the figure below.



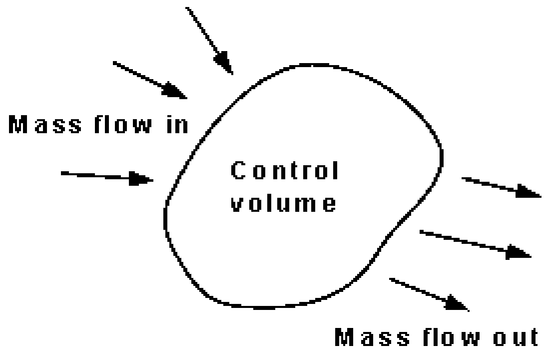
A typical velocity profile across a pipe

This idea, that mean velocity multiplied by the area gives the discharge, applies to all situations - not just pipe flow.

4. Continuity

Matter cannot be created or destroyed - (it is simply changed in to a different form of matter). This principle is know as the *conservation of mass* and we use it in the analysis of flowing fluids.

The principle is applied to fixed volumes, known as control volumes (or surfaces), like that in the figure below:



An arbitrarily shaped control volume.

For any control volume the principle of *conservation of mass* says

---

Mass entering per unit time = Mass leaving per unit time + Increase of mass in the control volume per unit time

---

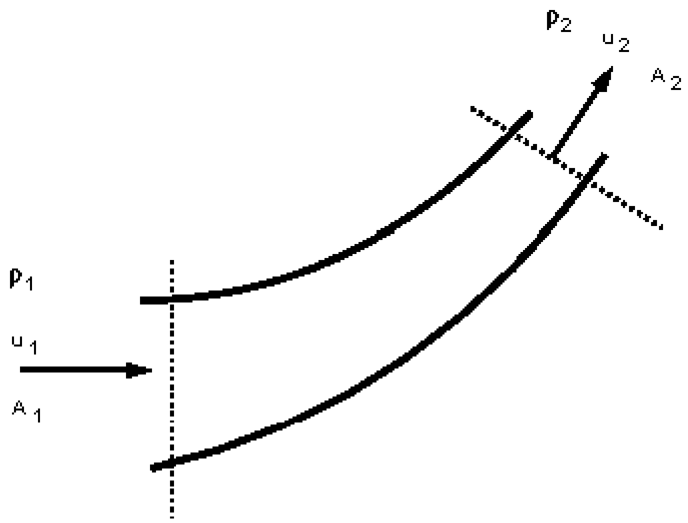
For **steady** flow there is no increase in the mass within the control volume, so

---

For steady flow

Mass entering per unit time = Mass leaving per unit time

This can be applied to a streamtube such as that shown below. No fluid flows across the boundary made by the streamlines so mass only enters and leaves through the two ends of this streamtube section.



A streamtube

We can then write

$$\text{mass entering per unit time at end 1} = \text{mass leaving per unit time at end 2}$$
$$\rho_1 \delta A_1 u_1 = \rho_2 \delta A_2 u_2$$

Or for steady flow,

$$\rho_1 \delta A_1 u_1 = \rho_2 \delta A_2 u_2 = \text{Constant} = \dot{m}$$

This is the equation of continuity.

The flow of fluid through a real pipe (or any other vessel) will vary due to the presence of a wall - in this case we can use the *mean* velocity and write

$$\rho_1 A_1 u_{m1} = \rho_2 A_2 u_{m2} = \text{Constant} = \dot{m}$$

When the fluid can be considered incompressible, i.e. the density does not change, $\rho_1 = \rho_2 = \rho$ so (dropping the *m* subscript)

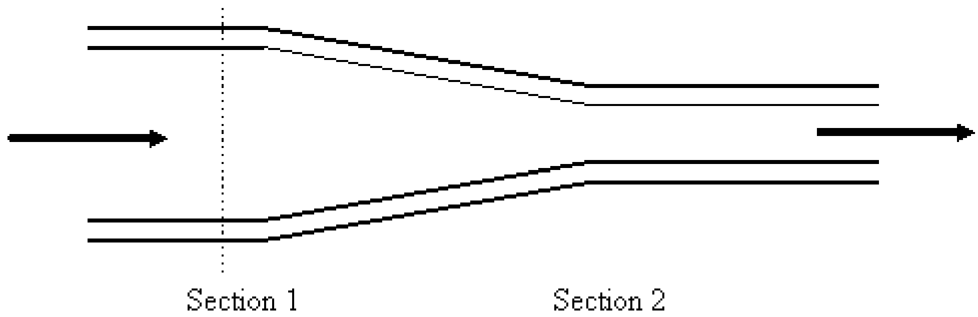
$$A_1 u_1 = A_2 u_2 = Q$$

This is the form of the continuity equation most often used.

This equation is a very powerful tool in fluid mechanics and will be used **repeatedly** throughout the rest of this course.

Some example applications

We can apply the principle of continuity to pipes with cross sections which change along their length. Consider the diagram below of a pipe with a contraction:



A liquid is flowing from left to right and the pipe is narrowing in the same direction. By the continuity principle, the *mass flow rate* must be the same at each section - the mass going into the pipe is equal to the mass going out of the pipe. So we can write:

$$A_1 u_1 \rho_1 = A_2 u_2 \rho_2$$

(with the sub-scripts 1 and 2 indicating the values at the two sections)

As we are considering a liquid, usually water, which is *not* very compressible, the density changes very little so we can say $\rho_1 = \rho_2 = \rho$. This also says that the *volume flow rate* is constant or that

$$\begin{aligned} \text{Discharge at section 1} &= \text{Discharge at section 2} \\ Q_1 &= Q_2 \\ A_1 u_1 &= A_2 u_2 \end{aligned}$$

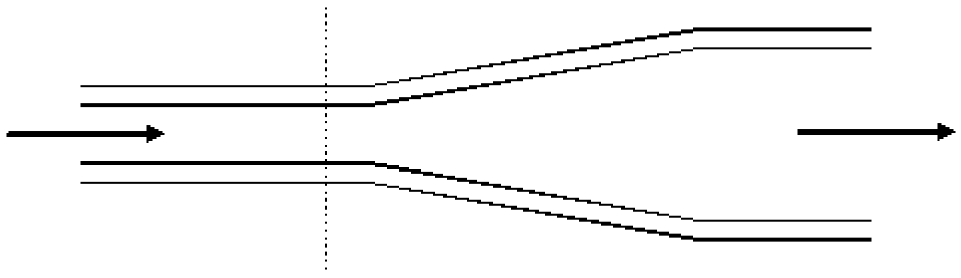
For example if the area $A_1 = 10 \times 10^{-3} m^2$ and $A_2 = 3 \times 10^{-3} m^2$ and the upstream mean velocity, $u_1 = 2.1 m/s$, then the downstream mean velocity can be calculated by

$$\begin{aligned} u_2 &= \frac{A_1 u_1}{A_2} \\ &= 7.0 m/s \end{aligned}$$

Notice how the downstream velocity only changes from the upstream by the ratio of the two areas of the pipe. As the area of the circular pipe is a function of the diameter we can reduce the calculation further,

$$u_2 = \frac{A_1}{A_2}u_1 = \frac{\pi d_1^2/4}{\pi d_2^2/4}u_1 = \frac{d_1^2}{d_2^2}u_1$$
$$= \left(\frac{d_1}{d_2}\right)^2 u_1$$

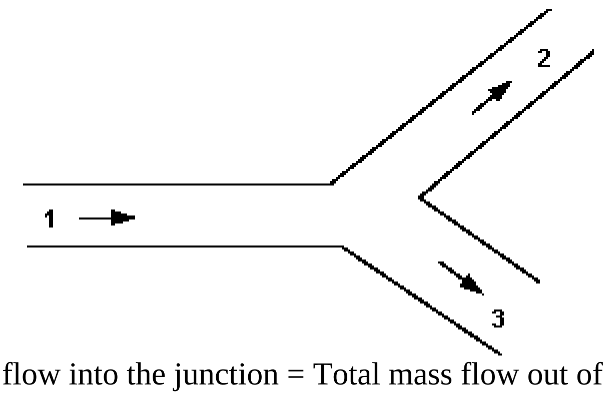
Now try this on a *diffuser*, a pipe which expands or diverges as in the figure below,



If the diameter at section 1 is $d_1 = 30mm$ and at section 2 $d_2 = 40mm$ and the mean velocity at section 2 is $u_2 = 3.0m/s$. The velocity entering the diffuser is given by,

$$u_1 = \left(\frac{40}{30}\right)^2 3.0$$
$$= 5.3m/s$$

Another example of the use of the continuity principle is to determine the velocities in pipes coming from a junction.



Total mass flow into the junction = Total mass flow out of the junction

$$\rho_1 Q_1 = \rho_2 Q_2 + \rho_3 Q_3$$

When the flow is incompressible (e.g. if it is water) $\rho_1 = \rho_2 = \rho$

$$Q_1 = Q_2 + Q_3$$
$$A_1 u_1 = A_2 u_2 + A_3 u_3$$

If pipe 1 diameter = 50mm, mean velocity 2m/s, pipe 2 diameter 40mm takes 30% of total discharge and pipe 3 diameter 60mm. What are the values of discharge and mean velocity in each pipe?

$$
\begin{aligned}
Q_1 &= A_1 u_1 = \left(\frac{\pi d^2}{4}\right) u \\
&= 0.00392 m^3/s \\
Q_2 &= 0.3Q_1 = 0.001178 m^3/s \\
Q_1 &= Q_2 + Q_3 \\
Q_3 &= Q_1 - 0.3Q_1 = 0.7Q_1 \\
&= 0.00275 m^3/s \\
Q_2 &= A_2 u_2 \\
u_2 &= 0.936 m/s \\
Q_3 &= A_3 u_3 \\
u_3 &= 0.972 m/s
\end{aligned}
$$

Go back to the main index page

Go back to the main index page

The Bernoulli equation

1. Work and energy

We know that if we drop a ball it accelerates downward with an acceleration $g = 9.81 m/s^2$ (neglecting the frictional resistance due to air). We can calculate the speed of the ball after falling a distance *h* by the formula $v^2 = u^2 + 2as$ (*a* = *g* and *s* = *h*). The equation could be applied to a falling droplet of water as the same laws of motion apply

A more general approach to obtaining the parameters of motion (of both solids and fluids) is to apply the principle of ***conservation of energy***. When friction is negligible the

---

sum of kinetic energy and gravitational potential energy is constant.

Kinetic energy $= \frac{1}{2} m v^2$

Gravitational potential energy $= mgh$

(*m* is the mass, *v* is the velocity and *h* is the height above the datum).

To apply this to a falling droplet we have an initial velocity of zero, and it falls through a height of h.

Initial kinetic energy $= 0$

Initial potential energy $= mgh$

Final kinetic energy $= \frac{1}{2}mv^2$

Final potential energy $= 0$

We know that

kinetic energy + potential energy = constant

so

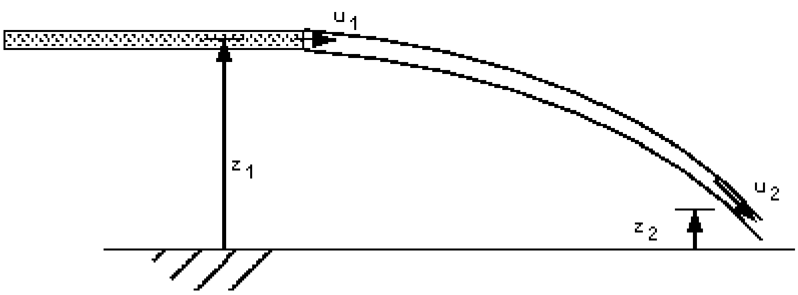
Initial kinetic energy + Initial potential energy = Final kinetic energy + Final potential energy

$$mgh = \frac{1}{2}mv^2$$

so

$$v = \sqrt{2gh}$$

Although this is applied to a drop of liquid, a similar method can be applied to a **continuous jet** of liquid.



The Trajectory of a jet of water

We can consider the situation as in the figure above - a continuous jet of water coming from a pipe with velocity u_1. One particle of the liquid with mass m travels

with the jet and falls from height z_1 to z_2. The velocity also changes from u_1 to u_2. The jet is travelling in air where the pressure is everywhere atmospheric so there is no force due to pressure acting on the fluid. The only force which is acting is that due to gravity. The sum of the kinetic and potential energies remains constant (as we neglect energy losses due to friction) so

$$mgz_1 + \frac{1}{2}mu_1^2 = mgz_2 + \frac{1}{2}mu_2^2$$

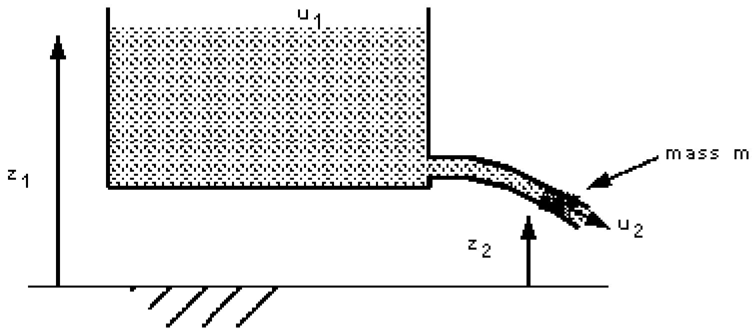
As m is constant this becomes

$$\frac{1}{2}u_1^2 + gz_1 = \frac{1}{2}u_2^2 + gz_2$$

This will give a reasonably accurate result as long as the weight of the jet is large compared to the frictional forces. It is only applicable while the jet is whole - before it breaks up into droplets.

Flow from a reservoir

We can use a very similar application of the energy conservation concept to determine the velocity of flow along a pipe from a reservoir. Consider the 'idealised reservoir' in the figure below.



An idealised reservoir

The level of the water in the reservoir is z_1. Considering the energy situation - there is no movement of water so kinetic energy is zero but the gravitational potential energy is mgz_1.

If a pipe is attached at the bottom water flows along this pipe out of the tank to a level z_2. A mass m has flowed from the top of the reservoir to the nozzle and it has gained a velocity u_2. The kinetic energy is now $\frac{1}{2}mu_2^2$ and the potential energy mgz_2. Summarising

Initial kinetic energy $= 0$

Initial potential energy $= mgz_1$

Final kinetic energy $= \frac{1}{2}mu_2^2$

Final potential energy $= mgz_2$

We know that

kinetic energy + potential energy = constant

so

$$mgz_1 = \frac{1}{2}mu_2^2 + mgz_2$$
$$mg(z_1 + z_2) = \frac{1}{2}mu_2^2$$

so

$$u_2 = \sqrt{2g(z_1 - z_2)}$$

We now have a expression for the velocity of the water as it flows from of a pipe nozzle at a height $(z_1 - z_2)$ below the surface of the reservoir. (Neglecting friction losses in the pipe and the nozzle).

Now apply this to this example: A reservoir of water has the surface at 310m above the outlet nozzle of a pipe with diameter 15mm. What is the a) velocity, b) the discharge out of the nozzle and c) mass flow rate. (Neglect all friction in the nozzle and the pipe).

$$\begin{aligned} u_2 &= \sqrt{2g(z_1 - z_2)} \\ &= \sqrt{2 \times g \times 310} \\ &= 78.0\, m/s \end{aligned}$$

Volume flow rate is equal to the area of the nozzle multiplied by the velocity

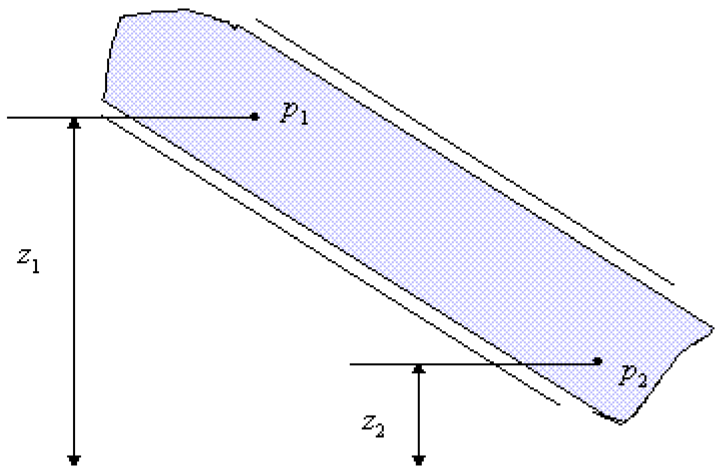
$$\begin{aligned} Q &= Au \\ &= (\pi \times \tfrac{0.015^2}{4}) \times 78.0 \\ &= 0.01378\, m^3/s \end{aligned}$$

The density of water is $1000 kg/m^3$ so the mass flow rate is

$$
\begin{aligned}
\text{mass flow rate} &= \text{density} \times \text{volume flow rate} \\
&= \rho Q \\
&= 1000 \times 0.01378 \\
&= 13.78\, kg/s
\end{aligned}
$$

In the above examples the resultant pressure force was always zero as the pressure surrounding the fluid was the everywhere the same - atmospheric. If the pressures had been different there would have been an extra force acting and we would have to take into account the work done by this force when calculating the final velocity.

We have already seen in the hydrostatics section an example of pressure difference where the velocities are zero.



The pipe is filled with stationary fluid of density ρ has pressures p_1 and p_2 at levels z_1 and z_2 respectively. What is the pressure difference in terms of these levels?

$$p_2 - p_1 = \rho g(z_1 - z_2)$$

or

$$\frac{p_1}{\rho} + gz_1 = \frac{p_2}{\rho} + gz_2$$

This applies when the pressure varies but the fluid is stationary.

Compare this to the equation derived for a moving fluid but constant pressure:

$$\frac{1}{2}u_1^2 + gz_1 = \frac{1}{2}u_2^2 + gz_2$$

You can see that these are similar form. What would happen if both pressure and velocity varies?

2. Bernoulli's Equation

Bernoulli's equation is one of the most important/useful equations in fluid mechanics. It may be written,

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

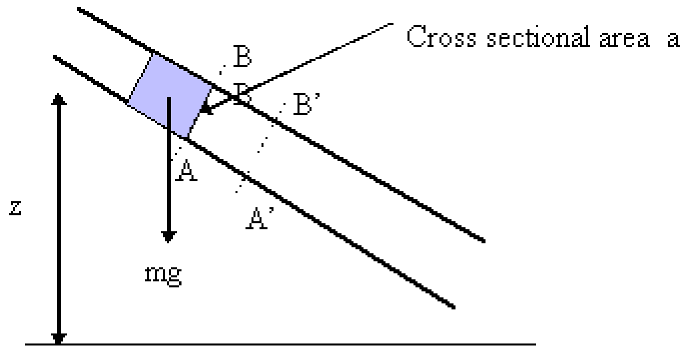
We see that from applying equal pressure or zero velocities we get the two equations from the section above. They are both just special cases of Bernoulli's equation.

Bernoulli's equation has some restrictions in its applicability, they are:

- Flow is steady;
- Density is constant (which also means the fluid is incompressible);
- Friction losses are negligible.
- The equation relates the states at two points along a single streamline, (not conditions on two different streamlines).

All these conditions are impossible to satisfy at any instant in time! Fortunately for many real situations where the conditions are *approximately* satisfied, the equation gives very good results.

The derivation of Bernoulli's Equation:



An element of fluid, as that in the figure above, has potential energy due to its height z above a datum and kinetic energy due to its velocity *u*. If the element has weight mg then

potential energy = mgz

---

potential energy per unit weight = z

---

kinetic energy = $\frac{1}{2}mu^2$

---

kinetic energy per unit weight = $\frac{u^2}{2g}$

At any cross-section the pressure generates a force, the fluid will flow, moving the cross-section, so work will be done. If the pressure at cross section AB is *p* and the area of the cross-section is *a* then

force on AB = pa

when the mass *mg* of fluid has passed AB, cross-section AB will have moved to A'B'

volume passing AB = $\frac{mg}{\rho g} = \frac{m}{\rho}$

therefore

distance AA' = $\frac{m}{\rho a}$

work done = force distance AA'

$= pa \times \frac{m}{\rho a} = \frac{pm}{\rho}$

---

work done per unit weight = $\frac{p}{\rho g}$

This term is know as the pressure energy of the flowing stream.

Summing all of these energy terms gives

$$\begin{array}{c}\text{Pressure} \\ \text{energy per} \\ \text{unit weight}\end{array} + \begin{array}{c}\text{Kinetic} \\ \text{energy per} \\ \text{unit weight}\end{array} + \begin{array}{c}\text{Potential} \\ \text{energy per} \\ \text{unit weight}\end{array} = \begin{array}{c}\text{Total} \\ \text{energy per} \\ \text{unit weight}\end{array}$$

or

$$\frac{p}{\rho g} + \frac{u^2}{2g} + z = H$$

As all of these elements of the equation have units of length, they are often referred to as the following:

pressure head = $\frac{p}{\rho g}$

velocity head = $\frac{u^2}{2g}$

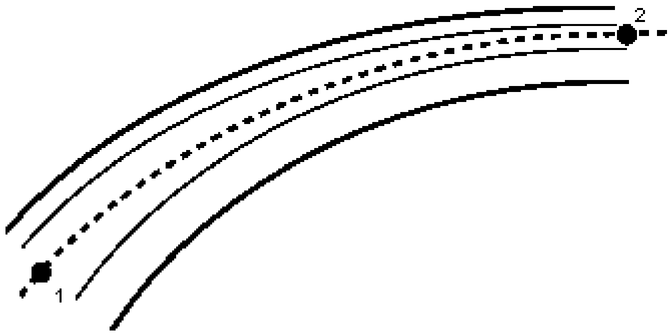
potential head = z

total head = H

By the principle of conservation of energy the total *energy* in the system does not change, Thus the total *head* does not change. So the Bernoulli equation can be written

$$\frac{p}{\rho g} + \frac{u^2}{2g} + z = H = \text{Constant}$$

As stated above, the Bernoulli equation applies to conditions along a streamline. We can apply it between two points, 1 and 2, on the streamline in the figure below



Two points joined by a streamline

total energy per unit weight at 1 = total energy per unit weight at 2

or

total head at 1 = total head at 2

or

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

This equation assumes no energy losses (e.g. from friction) or energy gains (e.g. from a pump) along the streamline. It can be expanded to include these simply, by adding the appropriate energy terms:

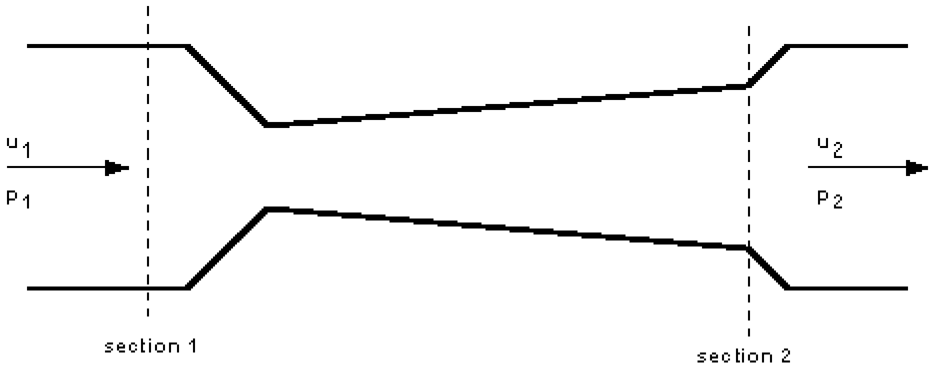
$$\begin{array}{ccccccccc} \text{Total} & & \text{Total} & & \text{Loss} & & \text{Work done} & & \text{Energy} \\ \text{energy per} & = & \text{energy per unit} & + & \text{per unit} & + & \text{per unit} & - & \text{supplied} \\ \text{unit weight at 1} & & \text{weight at 2} & & \text{weight} & & \text{weight} & & \text{per unit weight} \end{array}$$

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 + h + w - q$$

3. An example of the use of the Bernoulli equation.

When the Bernoulli equation is combined with the continuity equation the two can be used to find velocities and pressures at points in the flow connected by a streamline.

Here is an example of using the Bernoulli equation to determine pressure and velocity at within a contracting and expanding pipe.



A contracting expanding pipe

A fluid of constant density = 960 kg/m^3 is flowing steadily through the above tube. The diameters at the sections are $d_1 = 100mm$ and $d_2 = 80mm$. The gauge pressure at 1 is $p_1 = 200kN/m^2$ and the velocity here is $u_1 = 5m/s$. We want to know the gauge pressure at section 2.

We shall of course use the Bernoulli equation to do this and we apply it along a streamline joining section 1 with section 2.

The tube is horizontal, with $z_1 = z_2$ so Bernoulli gives us the following equation for pressure at section 2:

$$p_2 = p_1 + \frac{\rho}{2}(u_1^2 - u_2^2)$$

But we do not know the value of u_2. We can calculate this from the continuity equation: Discharge into the tube is equal to the discharge out i.e.

$$\begin{aligned} A_1 u_1 &= A_2 u_2 \\ u_2 &= \frac{A_1 u_1}{A_2} \\ u_2 &= \left(\frac{d_1}{d_2}\right)^2 u_1 \\ &= 7.8125 m/s \end{aligned}$$

So we can now calculate the pressure at section 2

$$\begin{aligned} p_2 &= 200000 - 17296.87 \\ &= 182703 \; N/m^2 \\ &= 182.7 \; kN/m^2 \end{aligned}$$

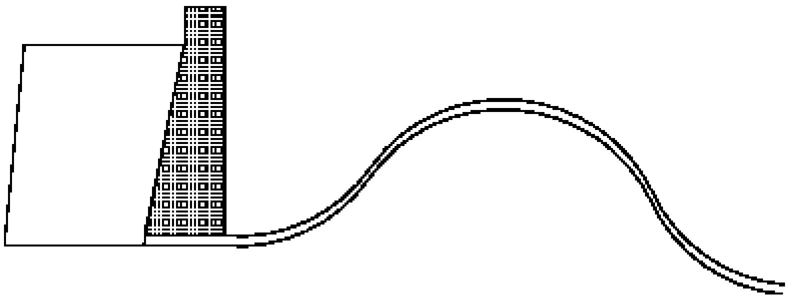
Notice how the velocity has increased while the pressure has decreased. The phenomenon - that pressure decreases as velocity increases - sometimes comes in very useful in engineering. (It is on this principle that carburettor in many car engines work - pressure reduces in a contraction allowing a small amount of fuel to enter).

Here we have used both the Bernoulli equation and the Continuity principle together to solve the problem. Use of this combination is very common. We will be seeing this again frequently throughout the rest of the course.

4. Pressure Head, Velocity Head, Potential Head and Total Head.

By looking again at the example of the reservoir with which feeds a pipe we will see how these different *heads* relate to each other.

Consider the reservoir below feeding a pipe which changes diameter and rises (in reality it may have to pass over a hill) before falling to its final level.



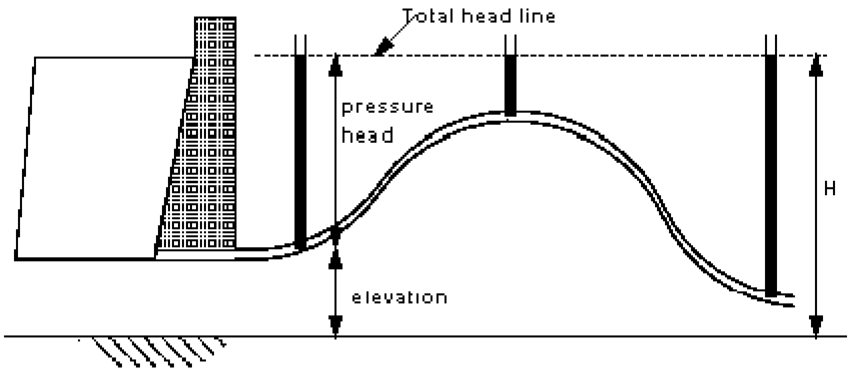
Reservoir feeding a pipe

To analyses the flow in the pipe we apply the Bernoulli equation along a streamline from point 1 on the surface of the reservoir to point 2 at the outlet nozzle of the pipe. And we know that the *total energy per unit weight* or the *total head* does not change - it is **constant** - along a streamline. But what is this value of this constant? We have the Bernoulli equation

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = H = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

We can calculate the total head, *H*, at the reservoir, $p_1 = 0$ as this is atmospheric and atmospheric gauge pressure is zero, the surface is moving very slowly compared to that in the pipe so $u_1 = 0$, so all we are left with is $total\,head = H = z_1$ the elevation of the reservoir.

A useful method of analysing the flow is to show the pressures graphically on the same diagram as the pipe and reservoir. In the figure above the *total head* line is shown. If we attached piezometers at points along the pipe, what would be their levels when the pipe nozzle was closed? (Piezometers, as you will remember, are simply open ended vertical tubes filled with the same liquid whose pressure they are measuring).



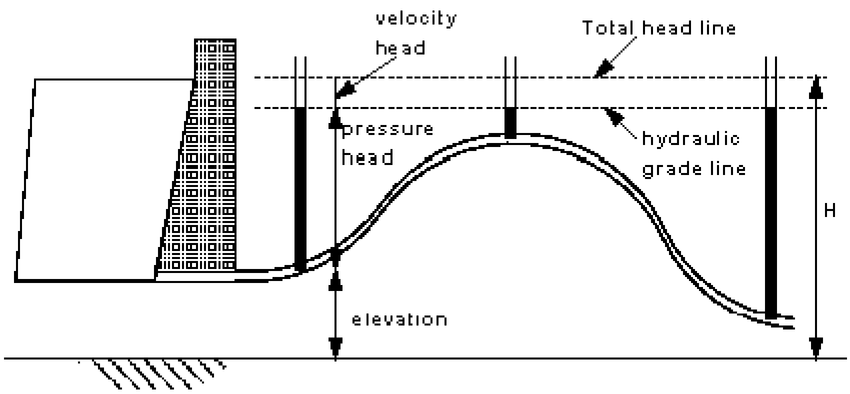
Piezometer levels with zero velocity

As you can see in the above figure, with zero velocity all of the levels in the piezometers are equal and the same as the total head line. At each point on the line, when $u = 0$

$$\frac{p}{\rho g} + z = H$$

The level in the piezometer is the *pressure head* and its value is given by $\frac{p}{\rho g}$.

What would happen to the levels in the piezometers (pressure heads) if the if water was flowing with velocity = *u*? We know from earlier examples that as velocity increases so pressure falls …

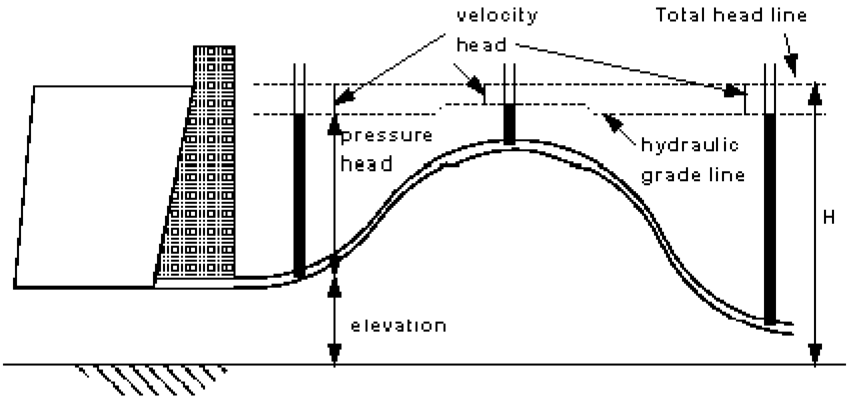


Piezometer levels when fluid is flowing

$$\frac{p}{\rho g} + \frac{u^2}{2g} + z = H$$

We see in this figure that the levels have reduced by an amount equal to the velocity head, $\frac{u^2}{2g}$. Now as the pipe is of constant diameter we know that the velocity is constant along the pipe so the velocity head is constant and represented graphically by the horizontal line shown. (this line is known as the *hydraulic grade line*).

What would happen if the pipe were not of constant diameter? Look at the figure below where the pipe from the example above is replaced be a pipe of three sections with the middle section of larger diameter



Piezometer levels and velocity heads with fluid flowing in varying diameter pipes

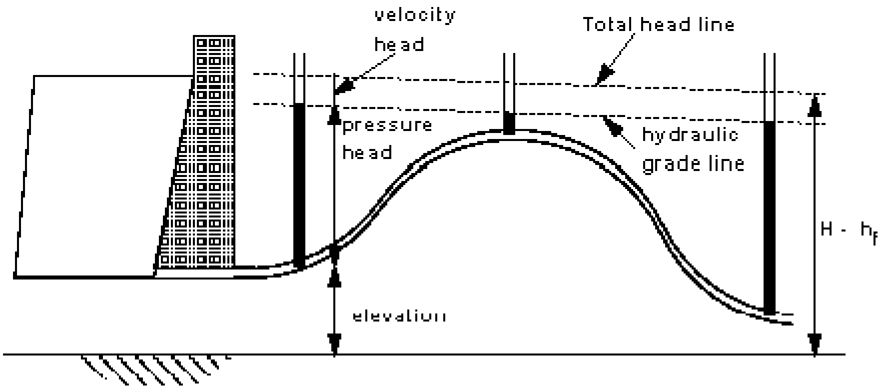
The velocity head at each point is now different. This is because the velocity is different at each point. By considering continuity we know that the velocity is different because the diameter of the pipe is different. Which pipe has the greatest diameter?

Pipe 2, because the velocity, and hence the velocity head, is the smallest.

This graphical representation has the advantage that we can see at a glance the pressures in the system. For example, where along the whole line is the lowest pressure head? It is where the hydraulic grade line is nearest to the pipe elevation i.e. at the highest point of the pipe.

Losses due to friction.

In a real pipe line there are energy losses due to friction - these must be taken into account as they can be very significant. How would the pressure and hydraulic grade lines change with friction? Going back to the constant diameter pipe, we would have a pressure situation like this shown below



Hydraulic Grade line and Total head lines for a constant diameter pipe with friction

How can the total head be changing? We have said that the total head - or total energy per unit weight - is constant. We are considering energy conservation, so if we allow for an amount of energy to be lost due to friction the total head will change. We have seen the equation for this before. But here it is again with the energy loss due to friction written as a *head* and given the symbol h_f . This is often know as the *head loss due to friction*.

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 + h_f$$

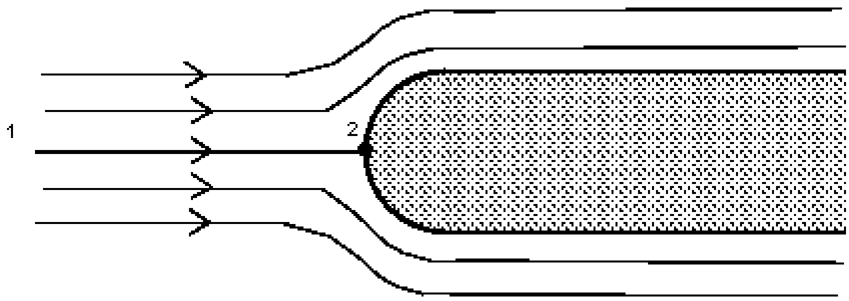
Go back to the main index page

Applications of the Bernoulli Equation

The Bernoulli equation can be applied to a great many situations not just the pipe flow we have been considering up to now. In the following sections we will see some examples of its application to flow measurement from tanks, within pipes as well as in open channels.

1. Pitot Tube

If a stream of uniform velocity flows into a blunt body, the stream lines take a pattern similar to this:



Streamlines around a blunt body

Note how some move to the left and some to the right. But one, in the centre, goes to the tip of the blunt body and stops. It stops because at this point the velocity is zero - the fluid does not move at this one point. This point is known as the *stagnation point*.

From the Bernoulli equation we can calculate the pressure at this point. Apply Bernoulli along the central streamline from a point upstream where the velocity is u_1 and the pressure p_1 to the stagnation point of the blunt body where the velocity is zero, $u_2 = 0$. Also $z_1 = z_2$.

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

$$\frac{p_1}{\rho} + \frac{u_1^2}{2} = \frac{p_2}{\rho}$$

$$p_2 = p_1 + \frac{1}{2}\rho u_1^2$$

This increase in pressure which bring the fluid to rest is called the *dynamic pressure*.

Dynamic pressure = $\frac{1}{2}\rho u_1^2$

or converting this to head (using $h = \frac{p}{\rho g}$)

Dynamic head = $\frac{1}{2g}u_1^2$

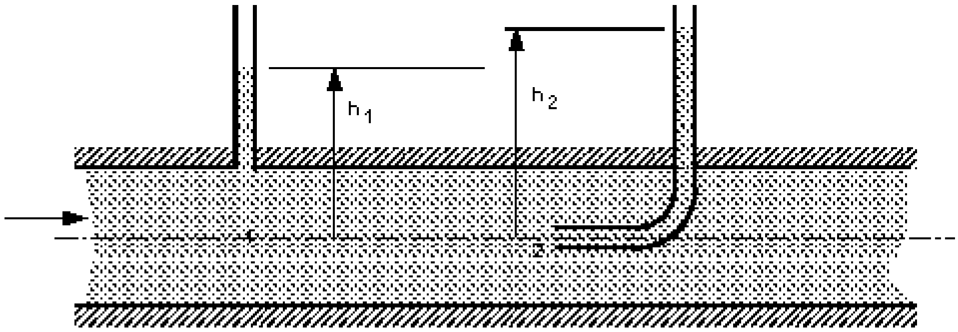
The total pressure is know as the *stagnation pressure* (or *total pressure*)

Stagnation pressure = $p_1 + \frac{1}{2}\rho u_1^2$

or in terms of head

Stagnation head = $\frac{p_1}{\rho g} + \frac{1}{2g}u_1^2$

The blunt body stopping the fluid does not have to be a solid. I could be a static column of fluid. Two piezometers, one as normal and one as a Pitot tube within the pipe can be used in an arrangement shown below to measure velocity of flow.



A Piezometer and a Pitot tube

Using the above theory, we have the equation for p_2 ,

$$p_2 = p_1 + \frac{1}{2}\rho u_1^2$$

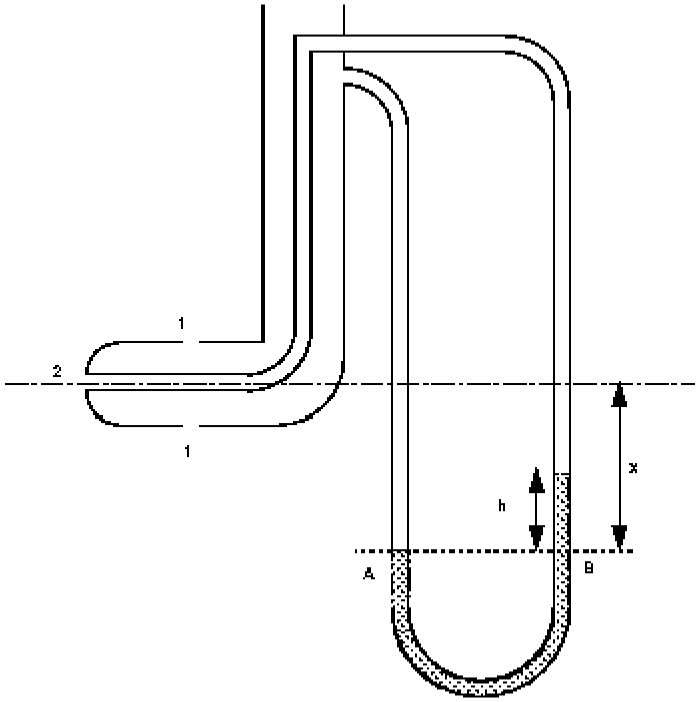
$$\rho g h_2 = \rho g h_1 + \frac{1}{2}\rho u_1^2$$

$$u = \sqrt{2g(h_2 - h_1)}$$

We now have an expression for velocity obtained from two pressure measurements and the application of the Bernoulli equation.

2. Pitot Static Tube

The necessity of two piezometers and thus two readings make this arrangement a little awkward. Connecting the piezometers to a manometer would simplify things but there are still two tubes. The *Pitot static* tube combines the tubes and they can then be easily connected to a manometer. A Pitot static tube is shown below. The holes on the side of the tube connect to one side of a manometer and register the *static head, (h_1)*, while the central hole is connected to the other side of the manometer to register, as before, the *stagnation head (h_2)*.



A Pitot-static tube

Consider the pressures on the level of the centre line of the Pitot tube and using the theory of the manometer,

$$
\begin{aligned}
p_A &= p_2 + \rho g X \\
p_B &= p_1 + \rho g(X - h) + \rho_{man} g h \\
p_A &= p_B \\
p_2 + \rho g X &= p_1 + \rho g(X - h) + \rho_{man} g h
\end{aligned}
$$

We know that $p_2 = p_{static} = p_1 + \frac{1}{2}\rho u_1^2$, substituting this in to the above gives

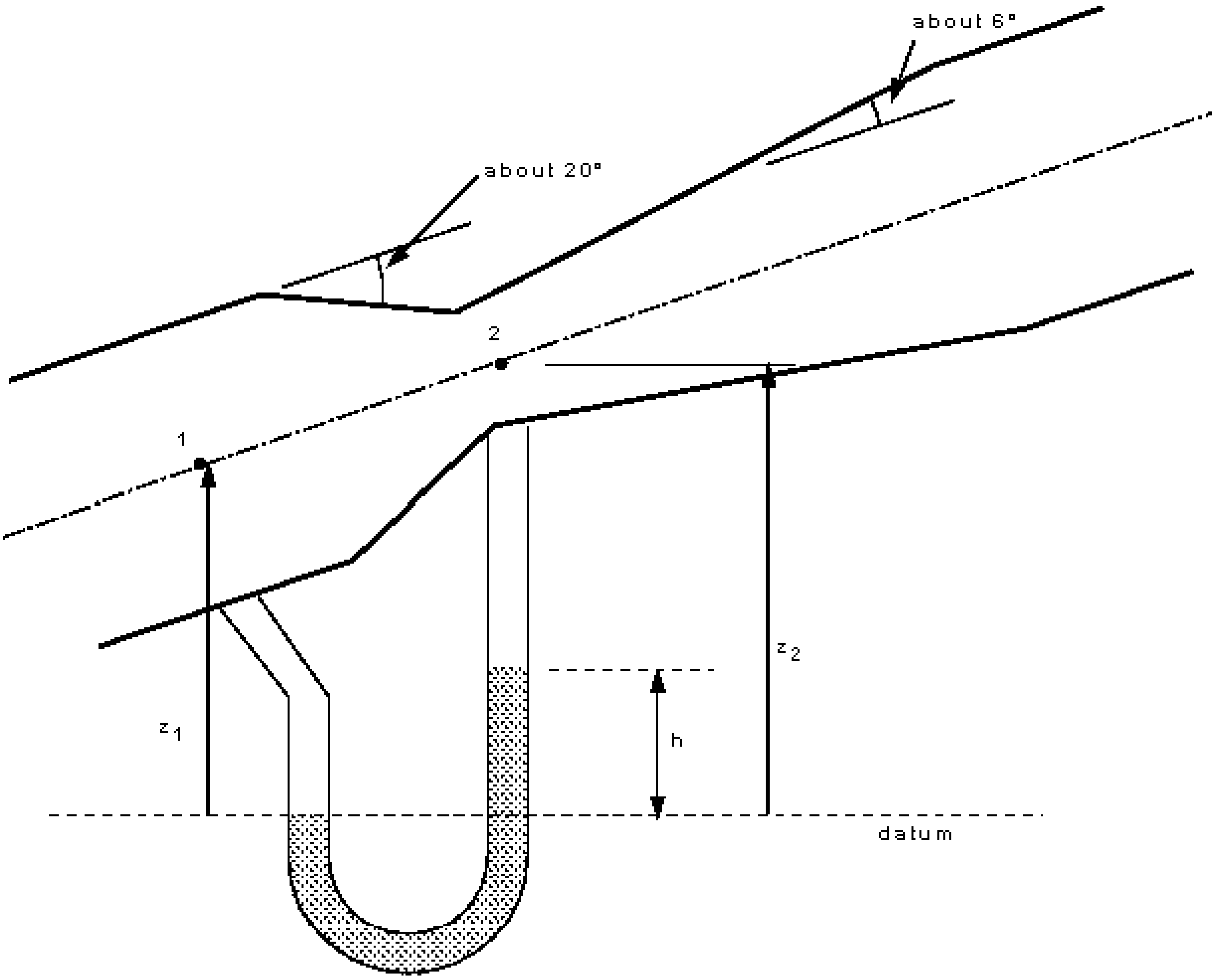
$$
\begin{aligned}
p_1 + hg(\rho_{man} - \rho) &= p_1 + \frac{\rho u_1^2}{2} \\
u_1 &= \sqrt{\frac{2gh(\rho_m - \rho)}{\rho}}
\end{aligned}
$$

The Pitot/Pitot-static tubes give velocities at points in the flow. It does not give the overall discharge of the stream, which is often what is wanted. It also has the drawback that it is liable to block easily, particularly if there is significant debris in the flow.

3. Venturi Meter

The Venturi meter is a device for measuring discharge in a pipe. It consists of a rapidly converging section which increases the velocity of flow and hence reduces the

pressure. It then returns to the original dimensions of the pipe by a gently diverging 'diffuser' section. By measuring the pressure differences the discharge can be calculated. This is a particularly accurate method of flow measurement as energy loss are very small.



A Venturi meter

Applying Bernoulli along the streamline from point 1 to point 2 in the narrow *throat* of the Venturi meter we have

$$\frac{p_1}{\rho g}+\frac{u_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{u_2^2}{2g}+z_2$$

By the using the continuity equation we can eliminate the velocity u_2,

$$Q=u_1A_1=u_2A_2$$
$$u_2=\frac{u_1A_1}{A_2}$$

Substituting this into and rearranging the Bernoulli equation we get

$$\frac{p_1 - p_2}{\rho g} + z_1 - z_2 = \frac{u_1^2}{2g}\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

$$u_1 = \sqrt{\frac{2g\left[\frac{p_1 - p_2}{\rho g} + z_1 - z_2\right]}{\left(\frac{A_1}{A_2}\right)^2 - 1}}$$

To get the theoretical discharge this is multiplied by the area. To get the actual discharge taking in to account the losses due to friction, we include a coefficient of discharge

$$Q_{ideal} = u_1 A_1$$
$$Q_{actual} = C_d Q_{ideal} = C_d u_1 A_1$$
$$Q_{actual} = C_d A_1 A_2 \sqrt{\frac{2g\left[\frac{p_1 - p_2}{\rho g} + z_1 - z_2\right]}{A_1^2 - A_2^2}}$$

This can also be expressed in terms of the manometer readings

$$p_1 + \rho g z_1 = p_2 + \rho_{man} g h + \rho g (z_2 - h)$$
$$\frac{p_1 - p_2}{\rho g} + z_1 - z_2 = h\left(\frac{\rho_{man}}{\rho} - 1\right)$$

Thus the discharge can be expressed in terms of the manometer reading::

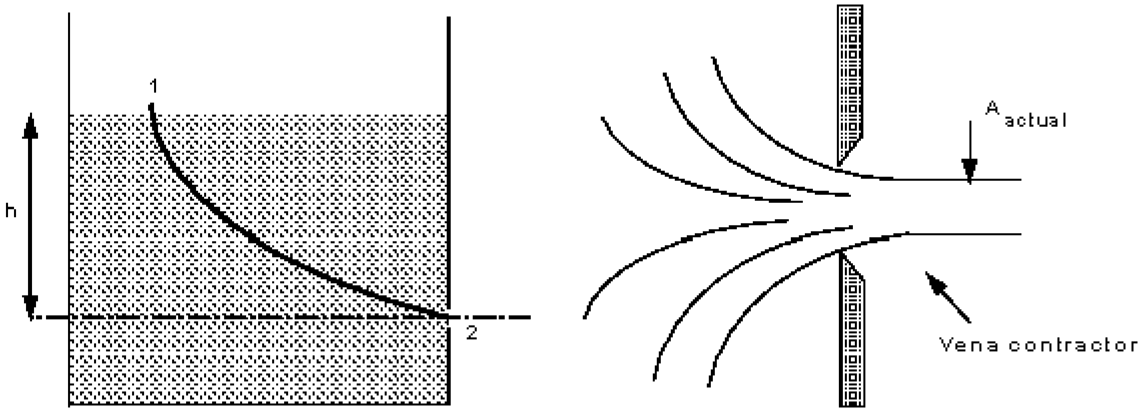
$$Q_{actual} = C_d A_1 A_2 \sqrt{\frac{2gh\left(\frac{\rho_{man}}{\rho} - 1\right)}{A_1^2 - A_2^2}}$$

Notice how this expression does not include any terms for the elevation or orientation (z_1 or z_2) of the Venturimeter. This means that the meter can be at any convenient angle to function.

The purpose of the diffuser in a Venturi meter is to assure gradual and steady deceleration after the throat. This is designed to ensure that the pressure rises again to something near to the original value before the Venturi meter. The angle of the diffuser is usually between 6 and 8 degrees. Wider than this and the flow might separate from the walls resulting in increased friction and energy and pressure loss. If the angle is less than this the meter becomes very long and pressure losses again become significant. The efficiency of the diffuser of increasing pressure back to the original is rarely greater than 80%.

4. Flow Through A Small Orifice

We are to consider the flow from a tank through a hole in the side close to the base. The general arrangement and a close up of the hole and streamlines are shown in the figure below



Tank and streamlines of flow out of the sharp edged orifice

The shape of the holes edges are as they are (sharp) to minimise frictional losses by minimising the contact between the hole and the liquid - the only contact is the very edge.

Looking at the streamlines you can see how they contract after the orifice to a minimum value when they all become parallel, at this point, the velocity and pressure are uniform across the jet. This convergence is called the *vena contracta*. (From the Latin 'contracted vein'). It is necessary to know the amount of contraction to allow us to calculate the flow.

We can predict the velocity at the orifice using the Bernoulli equation. Apply it along the streamline joining point 1 on the surface to point 2 at the centre of the orifice.

At the surface velocity is negligible ($u_1 = 0$) and the pressure atmospheric ($p_1 = 0$).At the orifice the jet is open to the air so again the pressure is atmospheric ($p_2 = 0$). If we take the datum line through the orifice then $z_1 = h$ and $z_2 = 0$, leaving

$$h = \frac{u_2^2}{2g}$$

$$u_2 = \sqrt{2gh}$$

This is the theoretical value of velocity. Unfortunately it will be an over estimate of the real velocity because friction losses have not been taken into account. To incorporate friction we use the **coefficient of velocity** to correct the theoretical velocity,

$$u_{actual} = C_v u_{theoretical}$$

Each orifice has its own coefficient of velocity, they usually lie in the range(0.97 - 0.99)

To calculate the discharge through the orifice we multiply the area of the jet by the velocity. The actual area of the jet is the area of the vena contracta **not** the area of the orifice. We obtain this area by using a **coefficient of contraction** for the orifice

$$A_{actual} = C_c A_{orifice}$$

So the discharge through the orifice is given by

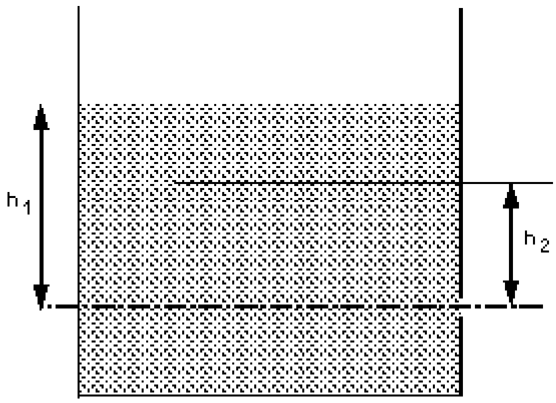
$$\begin{aligned} Q &= Au \\ Q_{actual} &= A_{actual} u_{actual} \\ &= C_c C_v A_{orifice} u_{theoretical} \\ &= C_d A_{orifice} u_{theoretical} \\ &= C_d A_{orifice} \sqrt{2gh} \end{aligned}$$

Where C_d is the **coefficient of discharge**, and $C_d = C_c C_v$

5. Time for a Tank to Empty

We now have an expression for the discharge out of a tank based on the height of water above the orifice. It would be useful to know how long it would take for the tank to empty.

As the tank empties, so the level of water will fall. We can get an expression for the time it takes to fall by integrating the expression for flow between the initial and final levels.



Tank emptying from level h_1 to h_2.

The tank has a cross sectional area of A. In a time dt the level falls by dh or the flow out of the tank is

$$Q = Av$$
$$Q = -A\frac{\delta h}{\delta t}$$

(-ve sign as δh is falling)

Rearranging and substituting the expression for *Q* through the orifice gives

$$\delta t = \frac{-A}{C_d A_o \sqrt{2g}} \frac{\delta h}{\sqrt{h}}$$

This can be integrated between the initial level, h_1, and final level, h_2, to give an expression for the time it takes to fall this distance

$$\begin{aligned} t &= \frac{-A}{C_d A_o \sqrt{2g}} \int_{h_1}^{h_2} \frac{\delta h}{\sqrt{h}} \\ &= \frac{-A}{C_d A_o \sqrt{2g}} \left[2\sqrt{h} \right]_{h_1}^{h_2} \\ &= \frac{-2A}{C_d A_o \sqrt{2g}} \left[\sqrt{h_2} - \sqrt{h_1} \right] \end{aligned}$$

1. Submerged Orifice

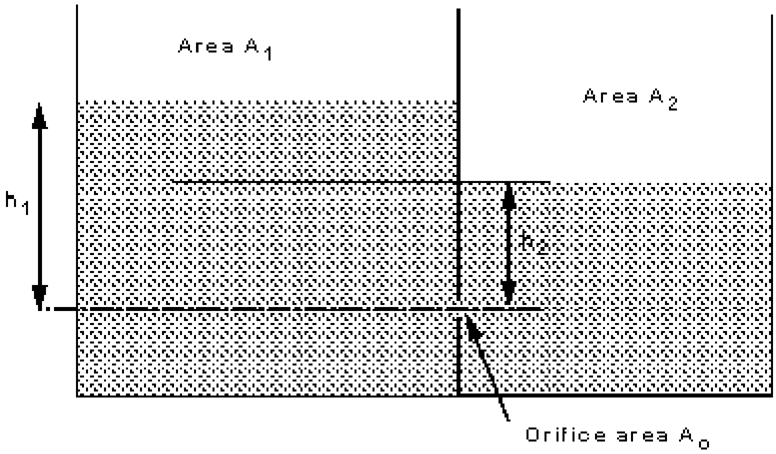
We have two tanks next to each other (or one tank separated by a dividing wall) and fluid is to flow between them through a submerged orifice. Although difficult to see, careful measurement of the flow indicates that the submerged jet flow behaves in a similar way to the jet in air in that it forms a vena contracta below the surface. To determine the velocity at the jet we first use the Bernoulli equation to give us the ideal velocity. Applying Bernoulli from point 1 on the surface of the deeper tank to point 2 at the centre of the orifice, gives

$$\begin{aligned} \frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 &= \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 \\ 0 + 0 + h_1 &= \frac{\rho g h_2}{\rho g} + \frac{u_2^2}{2g} + 0 \\ u_2 &= \sqrt{2g(h_1 - h_2)} \end{aligned}$$

i.e. the ideal velocity of the jet through the submerged orifice depends on the *difference* in head across the orifice. And the discharge is given by

$$\begin{aligned} Q &= C_d A_o u \\ &= C_d A_o \sqrt{2g(h_1 - h_2)} \end{aligned}$$

6. Time for Equalisation of Levels in Two Tanks



Two tanks of initially different levels joined by an orifice

By a similar analysis used to find the time for a level drop in a tank we can derive an expression for the change in levels when there is flow between two connected tanks.

Applying the continuity equation

$$Q = -A_1 \frac{\delta h_1}{\delta t} = A_2 \frac{\delta h_2}{\delta t}$$
$$Q\,\delta t = -A_1 \delta h_1 = A_2 \delta h_2$$

Also we can write $-\delta h_1 + \delta h_2 = \delta h$

So

$$-A_1 \delta h_1 = A_2 \delta h_1 - A_2 \delta h$$
$$\delta h_1 = \frac{A_2 \delta h}{A_1 + A_2}$$

Then we get

$$Q\,\delta t = -A_1 \delta h_1$$
$$C_d A_o \sqrt{2g(h_1 - h_2)}\ \delta t = \frac{A_1 A_2}{A_1 + A_2} \delta h$$

Re arranging and integrating between the two levels we get

$$\delta t = \frac{A_1 A_2}{(A_1 + A_2) C_d A_o \sqrt{2g}} \frac{\delta h}{\sqrt{h}}$$

$$t = \frac{A_1 A_2}{(A_1 + A_2) C_d A_o \sqrt{2g}} \int_{h_{initial}}^{h_{final}} \frac{\delta h}{\sqrt{h}}$$

$$= \frac{2 A_1 A_2}{(A_1 + A_2) C_d A_o \sqrt{2g}} \left[\sqrt{h} \right]_{h_{initial}}^{h_{final}}$$

$$= \frac{2 A_1 A_2}{(A_1 + A_2) C_d A_o \sqrt{2g}} \left[\sqrt{h_{initial}} - \sqrt{h_{final}} \right]$$

(remember that h in this expression is the *difference* in height between the two levels ($h_2 - h_1$) to get the time for the levels to equal use $h_{initial} = h_1$ and $h_{final} = 0$).

Thus we have an expression giving the time it will take for the two levels to equal.

Flow Over Notches and Weirs

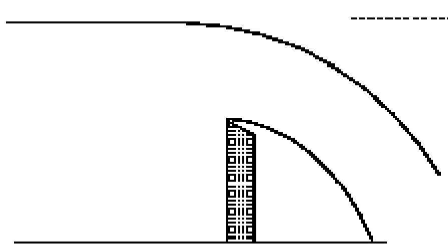
A notch is an opening in the side of a tank or reservoir which extends above the surface of the liquid. It is usually a device for measuring discharge. A weir is a notch on a larger scale - usually found in rivers. It may be sharp crested but also may have a substantial width in the direction of flow - it is used as both a flow measuring device and a device to raise water levels.

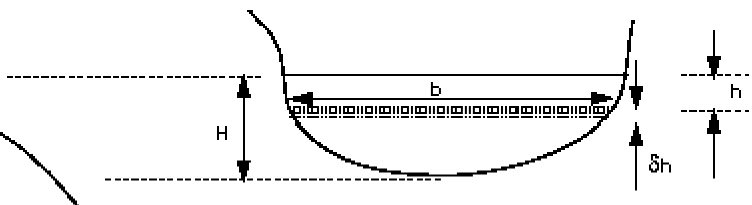
7. Weir Assumptions

We will assume that the velocity of the fluid approaching the weir is small so that kinetic energy can be neglected. We will also assume that the velocity through any elemental strip depends only on the depth below the free surface. These are acceptable assumptions for tanks with notches or reservoirs with weirs, but for flows where the velocity approaching the weir is substantial the kinetic energy must be taken into account (e.g. a fast moving river).

8. A General Weir Equation

To determine an expression for the theoretical flow through a notch we will consider a horizontal strip of width b and depth h below the free surface, as shown in the figure below.





Elemental strip of flow through a notch

$$\text{velocity through the strip}, \quad u = \sqrt{2gh}$$

$$\text{discharge through the strip}, \quad \delta Q = Au = b\,\delta h\sqrt{2gh}$$

integrating from the free surface, $h = 0$, to the weir crest, $h = H$ gives the expression for the total theoretical discharge

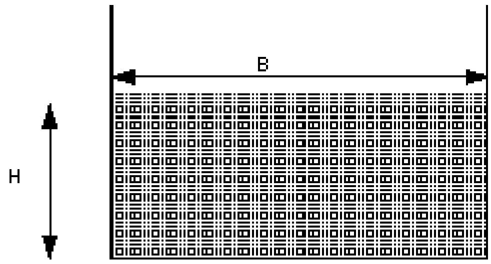
$$Q_{\text{theoretical}} = \sqrt{2g}\int_0^H b h^{1/2}\, dh$$

This will be different for every differently shaped weir or notch. To make further use of this equation we need an expression relating the width of flow across the weir to the depth below the free surface.

9. Rectangular Weir

For a rectangular weir the width does not change with depth so there is no relationship between *b* and depth *h*. We have the equation,

$$b = \text{constant} = B$$



A rectangular weir

Substituting this into the general weir equation gives

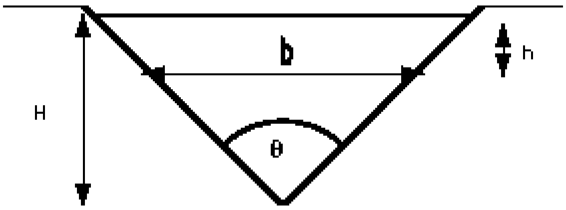
$$\begin{aligned} Q_{\text{theoretical}} &= B\sqrt{2g}\int_0^H h^{1/2}\, dh \\ &= \frac{2}{3}B\sqrt{2g}H^{3/2} \end{aligned}$$

To calculate the actual discharge we introduce a coefficient of discharge, C_d, which accounts for losses at the edges of the weir and contractions in the area of flow, giving

$$Q_{\text{actual}} = C_d \frac{2}{3}B\sqrt{2g}H^{3/2}$$

10. 'V' Notch Weir

For the "V" notch weir the relationship between width and depth is dependent on the angle of the "V".



"V" notch, or triangular, weir geometry.

If the angle of the "V" is θ then the width, *b*, a depth *h* from the free surface is

$$b = 2(H-h)\tan\left(\frac{\theta}{2}\right)$$

So the discharge is

$$\begin{aligned} Q_{\text{theoretical}} &= 2\sqrt{2g}\,\tan\left(\frac{\theta}{2}\right)\int_0^H (H-h)h^{1/2}\,dh \\ &= 2\sqrt{2g}\,\tan\left(\frac{\theta}{2}\right)\left[\frac{2}{5}Hh^{3/2} - \frac{2}{5}h^{5/2}\right]_0^H \\ &= \frac{8}{15}\sqrt{2g}\,\tan\left(\frac{\theta}{2}\right)H^{5/2} \end{aligned}$$

And again, the actual discharge is obtained by introducing a coefficient of discharge

$$Q_{\text{actual}} = C_d\,\frac{8}{15}\sqrt{2g}\,\tan\left(\frac{\theta}{2}\right)H^{5/2}$$

Go back to the main index page

Go back to the main index page
