Tikrit University

College of Engineering

Civil Engineering Department

M.Sc. Structure- 2013

Advanced of Solid Mechanics

M.Sc course 2013

Ass. Proof Dr.Muyasser M. Jomaa'h

B.Sc. in Civil Engineering   Ph.D in Structure Engineering

M.Sc Student

Waleed Abed Jasim

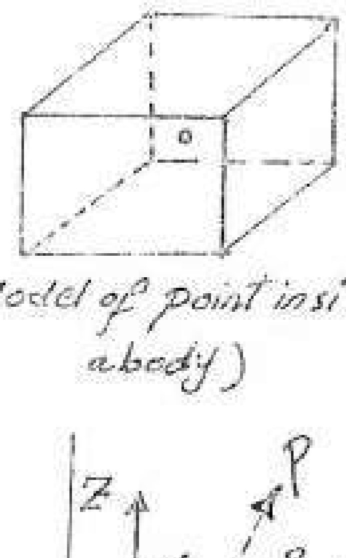
B.Sc. in Civil Engineering

عبدالرحمن عبدالله حمد

Stresses at a point inside a body (in x, y & z):

Imagine a small rectangular block enclosing the point. The block has "6" faces & on each face there are "3" stresses (one normal & two shearing stresses). Thus, $6 * 3 = 18$ stresses.

Normal stresses act in twos. Thus there are only "3" normal stresses (σ_x, σ_y & σ_z). Shearing stresses act in fours. Thus there are "3" shearing stresses (τ_{xy}, τ_{yz} & τ_{zx}). In 3-Dimensions, there are "6" stresses on the block.



(Model of point inside a body)

Direction Cosines:

Consider a line $\overline{OP}$ in x, y & z-axes. The direction of this line is specified by:

$l = \cos\alpha$ (with x-axis)

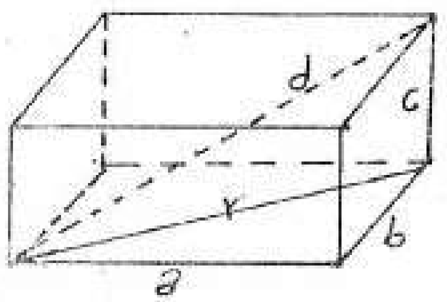
$m = \cos\beta$ (with y-axis)

$n = \cos\gamma$ (with z-axis)

Also, here $l^2 + m^2 + n^2 = 1$

τ_{xy}

To prove this, consider a diagonal through a rectangular block.



Notice that; $d^2 = c^2 + r^2$

& $r^2 = b^2 + a^2$

Then;

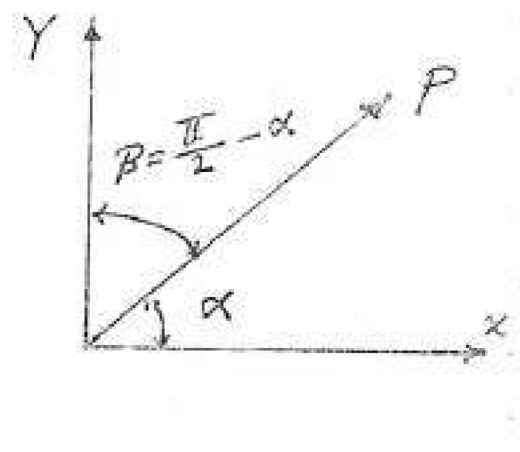
$$d^2 = c^2 + b^2 + a^2$$

or $$1 = \left(\frac{a}{d}\right)^2 + \left(\frac{b}{d}\right)^2 + \left(\frac{c}{d}\right)^2$$

$$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$= l^2 + m^2 + n^2$$

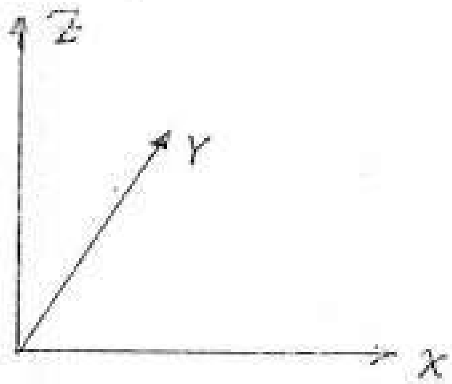
In 2-D, the angle $\gamma = 90^\circ$, thus $n = \cos\gamma = 0$ then; $l^2 + m^2 = 1$ or $\cos^2\alpha + \cos^2\beta = 1$ or ($\beta = 90^\circ - \alpha$).

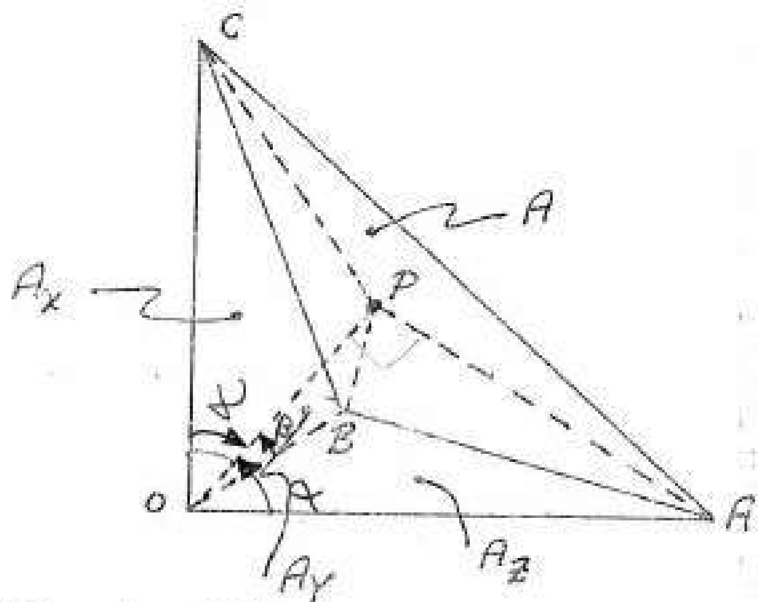


<u>Area of an inclined section:</u>

"A" is area of the inclined section. Let $n = \overline{OP}$ be the normal to "A".

<u>(3-D coordinates)</u>





Volume pyramid:

volume pyramid	$\overline{O}\ \overline{ABC} = \frac{1}{3} \cdot A \cdot \overline{OP}$
" "	$\overline{A}\ \overline{OBC} = \frac{1}{3} A_x \cdot \overline{OA}$
" "	$\overline{B}\ \overline{OAC} = \frac{1}{3} A_y \cdot \overline{OB}$
" "	$\overline{C}\ \overline{OBA} = \frac{1}{3} A_z \cdot \overline{OC}$

Then $A \cdot \overline{OP} = A_x \cdot \overline{OA} = A_y \cdot \overline{OB} = A_z \cdot \overline{OC}$

This gives;

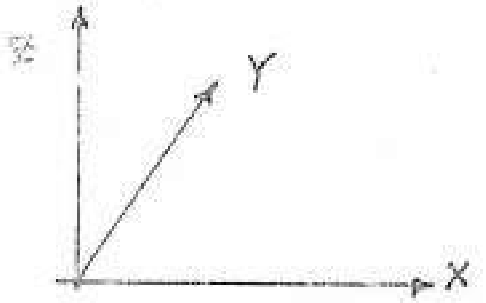
$$A_x = \frac{\overline{OP}}{\overline{OA}} \cdot A = \cos\alpha \cdot A = l \cdot A \Rightarrow l = \frac{A_x}{A}$$

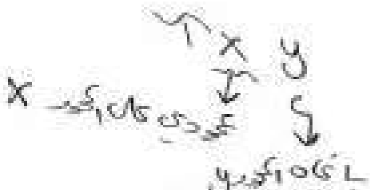
$$A_y = \frac{\overline{OP}}{\overline{OB}} \cdot A = \cos\beta \cdot A = m \cdot A \Rightarrow m = \frac{A_y}{A}$$

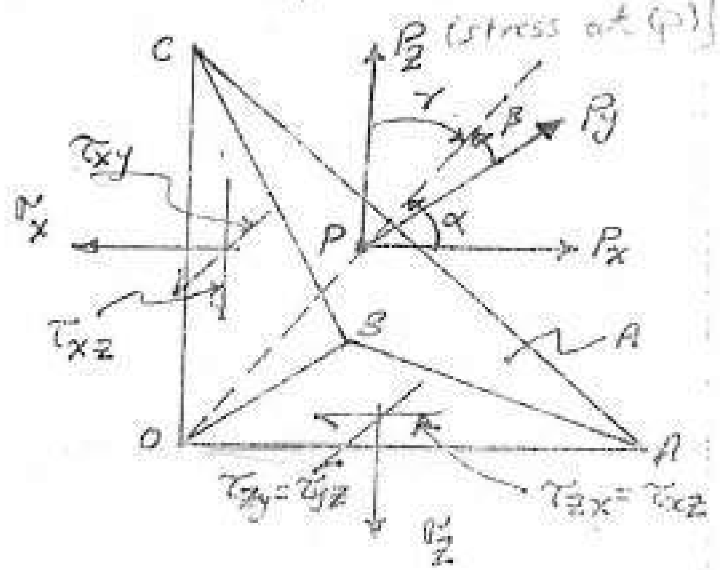
$$A_z = \frac{\overline{OP}}{\overline{OC}} \cdot A = \cos\gamma \cdot A = n \cdot A \Rightarrow n = \frac{A_z}{A}$$

stresses on an inclined surface:-

Consider a section through a rectangular block (in x, y & z).







Let $\overline{ABC}$ be the inclined section of area "A" & let P_x, P_y & P_z be the stresses in x, y & z - direction on this section.

Find P_x, P_y & P_z from equilibrium of forces, first take for x-direction:

$$P_x \cdot A = \sigma_x \cdot A_x + \tau_{yx} \cdot A_y + \tau_{zx} \cdot A_z$$

Then,

$$P_x = \sigma_x \cdot \frac{A_x}{A} + \tau_{yx} \cdot \frac{A_y}{A} + \tau_{zx} \cdot \frac{A_z}{A}$$

or,

$$P_x = l \cdot \sigma_x + m \cdot \tau_{yx} + n \cdot \tau_{zx}$$

For y-direction: $P_y = l \cdot \tau_{xy} + m \cdot \sigma_y + n \cdot \tau_{zy}$

For z-direction: $P_z = l \cdot \tau_{xz} + m \cdot \tau_{yz} + n \cdot \sigma_z$

In matrix:
$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

The normal stress "σ_n" on the inclined section will be:

$$\sigma_n = P_x \cdot \cos\alpha + P_y \cdot \cos\beta + P_z \cdot \cos\gamma$$
$$\sigma_n = l \cdot P_x + m \cdot P_y + n \cdot P_z$$
$$= l \cdot (l \cdot \sigma_x + m\,\tau_{xy} + n \cdot \tau_{xz}) + m \cdot (l \cdot \tau_{yx} + m \cdot \sigma_y + n \cdot \tau_{yz}) + n \cdot (l \cdot \tau_{zx} + m \cdot \tau_{zy} + n \cdot \sigma_z)$$

or $$\sigma_n = l^2 \cdot \sigma_x + m^2 \cdot \sigma_y + n^2 \cdot \sigma_z + 2lm \cdot \tau_{xy} + 2mn\,\tau_{yz} + 2nl \cdot \tau_{zx}$$

In matrix: $$\sigma_n = [\,l \;\; m \;\; n\,] \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$
$$= [\,l \; m \; n\,] \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

The shearing stress τ_{nt} is derived from:-

$$\tau_{nt}^2 + \sigma_n^2 = P_x^2 + P_y^2 + P_z^2 \quad \text{(same resultant)}$$

Thus, $$\tau_{nt} = \sqrt{(P_x^2 + P_y^2 + P_z^2) - \sigma_n^2}$$

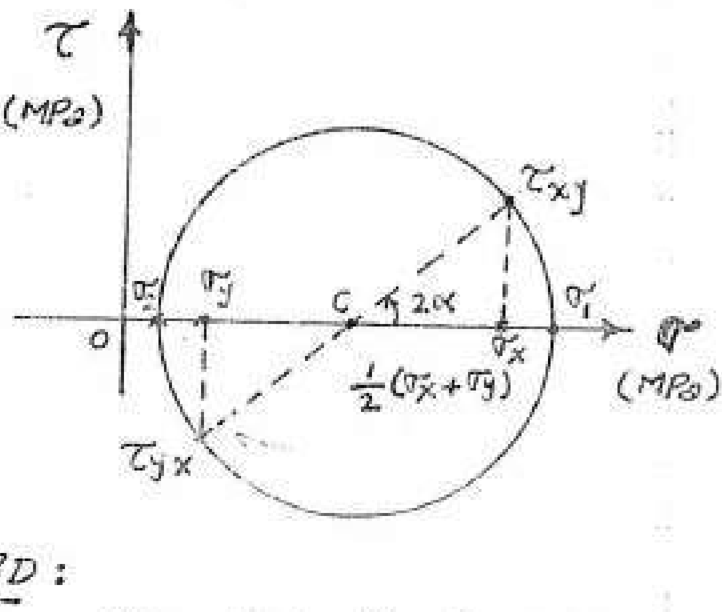
Notice, in two dimensions (x & y), Then $n = \cos\gamma$

$= \cos\frac{\pi}{2} = 0$, then;

$$\sigma_n = l^2 \cdot \sigma_x + m^2 \cdot \sigma_y + 2 l m . \tau_{xy}$$

$$= \cos^2\alpha . \sigma_x + (\cos^2\beta = \sin^2\alpha) . \sigma_y + 2\cos\alpha . \sin\alpha . \tau_{xy}$$

Mohr's Circle:

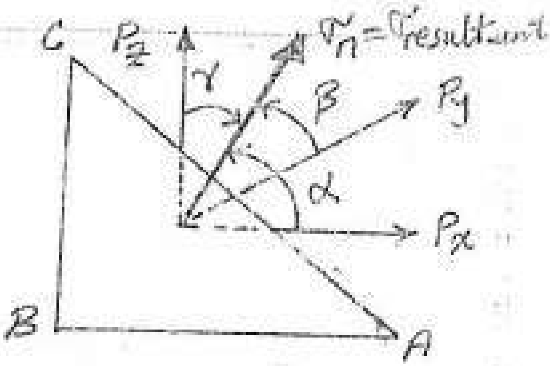


Principal stresses in 3D:

A principal stress is a normal stress on a plane without shearing stress.

Consider a rectangular block under stresses (σ_x, ..., τ_{zx}). search for the principal stresses & their directions. Return to the stresses on an inclined section. If "σ" is a principal stress on this section ($\sigma_n = \sigma$ & $\tau_{nt} = 0$), Then, (by resolution of forces):-

$P_x = \sigma \cdot \cos\alpha = l \cdot \sigma$
$P_y = \sigma \cdot \cos\beta = m \cdot \sigma$
$P_z = \sigma \cdot \cos\gamma = n \cdot \sigma$



Then;

$$\begin{cases} l \cdot \sigma_x + m \cdot \tau_{xy} + n \cdot \tau_{xz} = l \cdot \sigma \\ l \cdot \tau_{yx} + m \cdot \sigma_y + n \cdot \tau_{yz} = m \cdot \sigma \\ l \cdot \tau_{zx} + m \cdot \tau_{zy} + n \cdot \sigma_z = n \cdot \sigma \end{cases}$$

Here l, m & n are unknowns, also "σ" is unknown.

Rewrite;

$$l \cdot \overbrace{(\sigma_x - \sigma)}^{\to\ \text{constant}} + m \cdot \tau_{xy} + n \cdot \tau_{xz} = 0$$
$$l \cdot \tau_{yx} + m\,(\sigma_y - \sigma) + n \cdot \tau_{yz} = 0$$
$$l \cdot \tau_{zx} + m \cdot \tau_{zy} + n(\sigma_z - \sigma) = 0$$

These are "3" equations in "3" unknowns (l, m & n).

For nontrivial solution:

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_y - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_z - \sigma) \end{vmatrix} = 0$$

(EIGEN VALUE)

Expand & get a cubic equation in "σ". The 3 roots are σ_1, σ_2 & σ_3. The magnitudes ~~be~~ ~~the~~ are obtained. Next find their directions. Let

1. A rectangular block has the following stresses: $\sigma_x = 20\,MPa$, $\sigma_y = -12$, $\sigma_z = -2$, $\tau_{xy} = 10$, $\tau_{yz} = -6$, $\tau_{zx} = 5$. Determine the principal stresses & their directions?

ans.. Let σ be principal stress. Then:

$$\begin{vmatrix} (20-\sigma) & 10 & 5 \\ 10 & (-12-\sigma) & -6 \\ 5 & -6 & (-2-\sigma) \end{vmatrix} = 0$$

Expand: $$(20-\sigma)\begin{vmatrix} (-12-\sigma) & -6 \\ -5 & (-2-\sigma) \end{vmatrix} - 10\begin{vmatrix} 10 & -6 \\ 5 & (-2-\sigma) \end{vmatrix}$$

$$+5\begin{vmatrix} 10 & (-12-\sigma) \\ 5 & -6 \end{vmatrix} = 0$$

or $$(20-\sigma)\left[(-12-\sigma)(-2-\sigma) - 36\right] - 10\left[10(-2-\sigma) + 30\right]$$

$$+ 5\left[-60 - 5(-12-\sigma)\right] = 0$$

simplify:- $\sigma^3 - 6\sigma^2 - 417\sigma + 340 = 0$ [illegible]

[illegible] etc

Solve $\sigma_1 = 23.285\,MPa$, $\sigma_2 = 0.807$, $\sigma_3 = -18.090$

* Next find the direction cosines of the normals to the planes of principal stresses.
* start with $\sigma_1 = 23.285\,MPa$. let $l_1, m_1, \& n_1$ be the direction cosines. Return to:

$(20-23.285)l_1+10m_1+5n_1=0$
$10l_1+(-12-23.285)m_1-6n_1=0$
$5l_1-6m_1+(-\cancel{2}-23.285)n_1=0$

[illegible]

Take any two & find m_1 & n_1 in terms of l_1 substitute into $l_1^2+m_1^2+n_1^2=1$.

The results are: $l_1=0.9588$
$m_1=0.2527$
$n_1=0.1303$

[illegible]

Repeat for $\sigma_2=0.8071\,MPa$. The results are:

$l_2=0.0108$
$m_2=0.4311$
$n_2=-0.902$

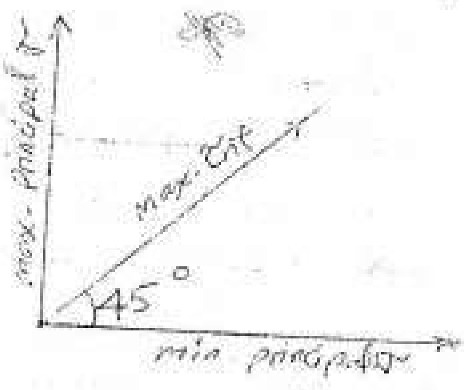
For $\sigma_3=-18.090\,MPa$: $l_3=-0.2814$, $m_3=0.8660$, $n_3=0.41$

Check orthogonality of the directions of principal stresses :-

$l_il_j+m_im_j+n_in_j=0 \quad$ for i or $j=1,2,3$ & $i\neq j$

$$\max \tau_{nt}=\frac{\max.\sigma_n-\min\sigma_n}{2}$$

[illegible]



(ii)

2. The stresses in a rectangular block are: $\sigma_x = 4\,N/mm^2$, $\sigma_y = 7$, $\sigma_z = -9$, $\tau_{xy} = 4$, $\tau_{yz} = 5$, $\tau_{zx} = 0$

i) Find the normal & shearing stresses on a section making $55°$ with x-axis & $60°$ with y-axis.

ii) Find the principal stresses & their directions.

iii) Find the maximum shearing stress in the block.

Solution:- i) use
$$P_x = \sigma_x . l + \tau_{xy} . m + \tau_{xz} . n$$
$$P_y = \tau_{yx} . l + \sigma_y . m + \tau_{yz} . n$$
$$P_z = \tau_{zx} . l + \tau_{zy} . m + \sigma_z . n$$

Then the normal stress (σ_n) on the inclined section is:

$\sigma_n = P_x . l + P_y . m + P_z . n$

or use; $\sigma_n = \sigma_x . l^2 + \sigma_y . m^2 + \sigma_z . n^2 + 2\tau_{xy}\, lm + 2\tau_{yz} . mn + 2\tau_{xz} . nl$

Here; $l = \cos\alpha = \cos 55° = 0.57357$

$m = \cos\beta = \cos 60° = 0.50$

$n = \sqrt{1 - l^2 - m^2} = 0.5805$

substitute:

$P_x = 4.57116\ N/m^2$

$P_y = 8.97301$

$P_z = -2.7233$

Also $\sigma_n = 5.34426\ N/m^2$

$\tau_{nt} = \sqrt{(P_x^2 + P_y^2 + P_z^2) - \sigma_n^2} = 8.64128\ N/m^2$

ii) For principal stresses, use
$$\begin{vmatrix} (4-\sigma) & 4 & 0 \\ 4 & (7-\sigma) & 5 \\ 0 & 5 & (-9-\sigma) \end{vmatrix} = 0$$

Expand & simplify; $\sigma^3 - 2\sigma^2 - 112\sigma + 208 = 0$

Sol:- $\sigma_1 = 10.67\ N/mm^2$, $\sigma_2 = 1.85$, $\sigma_3 = -10.52$

For $\sigma_1 = 10.67\ N/mm^2$: $l_1 = 0.553$, $m_1 = 0.838$, $n_1 = 0.21$

For $\sigma_2 = 1.85\ N/mm^2$: $l_2 = -0.8506$, $m_2 = 0.4626$, $n_2 = 0.2$

For $\sigma_3 = -10.52\ N/mm^2$: $l_3 = 0.280$, $m_3 = 0.290$, $n_3 = -0.94$

iii) Maximum shearing stress is:

$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min})$ = Radius of Mohr circle

Here:

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

$$= \frac{1}{2}(10.67 + 10.52) = 10.6\ N/mm^2$$

The plane of τ_{max} is $45°$ with the planes of σ_m & σ_{min}.

عبدالرحمن عبدالله محمد

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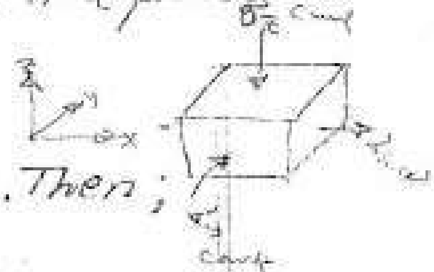
Comp. (−)
Tens. (+)

Examples (stresses):

1. A rectangular block has the following stresses: $\sigma_x = 20$ MPa, $\sigma_y = -12$ MPa و $\sigma_z = -2$ MPa, $\tau_{xy} = 10$ MPa, $\tau_{yz} = -6$ MPa & $\tau_{zx} = 5$ MPa, Determine the principal stresses & their directions?



Solution: Let (σ) be principal stress. Then;

$$\begin{vmatrix} (20-\sigma) & 10 & 5 \\ 10 & (-12-\sigma) & -6 \\ 5 & -6 & (-2-\sigma) \end{vmatrix} = 0$$

Expand: $(20-\sigma)\begin{vmatrix} (-12-\sigma) & -6 \\ -6 & (-2-\sigma) \end{vmatrix} - 10\begin{vmatrix} 10 & -6 \\ 5 & (-2-\sigma) \end{vmatrix}$

$$+5\begin{vmatrix} 10 & (-12-\sigma) \\ 5 & -6 \end{vmatrix} = 0$$

or $(20-\sigma)\left[(-12-\sigma)(-2-\sigma) - 36\right] - 10\left[10(-2-\sigma)\right.$

$$\left. +30\right] + 5\left[-60 - 5(-12-\sigma)\right] = 0$$

Simplify: $\sigma^3 - 6\sigma^2 - 417\sigma + 340 = 0$

Solve, $\sigma_1 = 23.285$ MPa, $\sigma_2 = 0.807$ MPa, $\sigma_3 = -18.09$ MPa

Next find the direction cosines of the normals to the lanes of principal stresses.

start with $\sigma_1 = 23.285$ MPa. let l_1, m_1 & n_1 e the direction cosines. Return to:

$$(20 - 23.285)\, l_1 + 10 m_1 + 5 n_1 = 0$$
$$10\, l_1 + (-12 - 23.285)\, m_1 - 6 n_1 = 0$$
$$5\, l_1 - 6 m_1 + (-2 - 23.285)\, n_1 = 0$$

take any two? bcs the third is repeated?

Take any two & find m_1 & n_1 in terms of l_1, substitute into;

$$l_1^2 + m_1^2 + n_1^2 = 1$$

The results are;

$$\left.\begin{aligned} l_1 &= 0.9588 \\ m_1 &= 0.2522 \\ n_1 &= 0.1303 \end{aligned}\right\}$$

all values less than 1 because bcs: $[\cos\alpha \le 1]$?

* Repeat for $\sigma_2 = 0.807$ MPa. The results are;

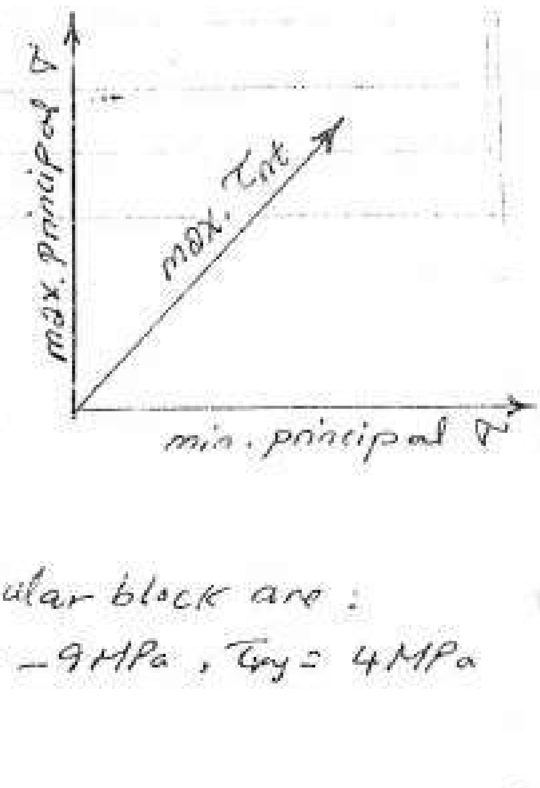
$l_2 = 0.0108$, $m_2 = 0.4311$, $n_2 = -0.902$

For $\sigma_3 = -18.090$ MPa : $l_3 = -0.2814$, $m_3 = 0.866$, $n_3 = 0.411$

check orthogonality of the directions of principal stresses:

$$l_i l_j + m_i m_j + n_i n_j = 0 \quad \text{For } i \text{ or } j = 1,2,3 \ \& \ i \neq j$$

$$\max \tau_{nt} = \frac{\max.\sigma_n - \min.\sigma_n}{2}$$



2. The stresses in a rectangular block are:
$\sigma_x = 4$ MPa, $\sigma_y = 7$ MPa, $\sigma_z = -9$ MPa, $\tau_{xy} = 4$ MPa, $\tau_{yz} = 5$ MPa & $\tau_{zx} = 0$

i) Find the normal & shearing stresses on a section making 50° with x-axis & 60° with y-axis.
ii) Find the principal stresses & their directions.
iii) Find the maximum shearing stress in the block.

solution:

i) use $P_x = \sigma_x \cdot l + \tau_{xy} \cdot m + \tau_{xz} \cdot n$

$$P_y = \tau_{yx} \cdot l + \sigma_y \cdot m + \tau_{yz} \cdot n$$

$$P_z = \tau_{zx} \cdot l + \tau_{zy} \cdot m + \sigma_z \cdot n$$

Then the normal stress σ_n on the inclined section is; $\sigma_n = P_x \cdot l + P_y \cdot m + P_z \cdot n$.

or use; $\sigma_n = \sigma_x \cdot l^2 + \sigma_y \cdot m^2 + \sigma_z \cdot n^2 + 2\tau_{xy} \cdot lm + 2\tau_{yz} \cdot mn + 2\tau_{xz} \cdot nl$

Here; $l = \cos\alpha = \cos 50^\circ = 0.64279$
$m = \cos\beta = \cos 60^\circ = 0.50$
$n = \sqrt{1 - l^2 - m^2} = 0.5805$

substitute: $P_x = 4.57116 \ N/m^2$
$P_y = 8.97301$
$P_z = -2.7233$

Also,

$\sigma_n = 5.84426 \ N/m^2$

$\tau_{nt} = \sqrt{(P_x^2 + P_y^2 + P_z^2) - \sigma_n^2} = 8.64128 \ N/m^2$

σ_1

(ii) For principal stresses, use;

$$\begin{vmatrix} (4-\sigma) & 4 & 0 \\ 4 & (7-\sigma) & 5 \\ 0 & 5 & (-9-\sigma) \end{vmatrix} = 0$$

Expand & simplify;

$$\sigma^3 - 2\sigma^2 - 112\sigma + 208 = 0$$

$-10.522 \quad\quad 10.66 \quad\quad 1.852$

Solve: $\sigma_1 = 10.67\ N/mm^2,\ \sigma_2 = 1.85,\ \sigma_3 = -10.52$

For $\sigma_1 = 10.67\ N/mm^2$; $l_1 = 0.553,\ m_1 = 0.838,\ n_1 = 0.218$
$\sigma_2 = 1.85\ N/mm^2$; $l_2 = -0.8606,\ m_2 = 0.4626,\ n_1 = 0.2132$
$\sigma_3 = -10.52\ N/mm^2$; $l_3 = 0.080,\ m_3 = 0.290,\ n_3 = -0.947$

(iii) Maximum shearing stress is:

$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \text{Radius of Mohr circle}$$

Here;

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

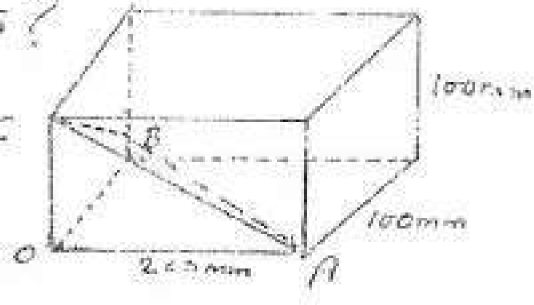
$$= \frac{1}{2}(10.67 + 10.52) = 10.6\ N/mm^2$$

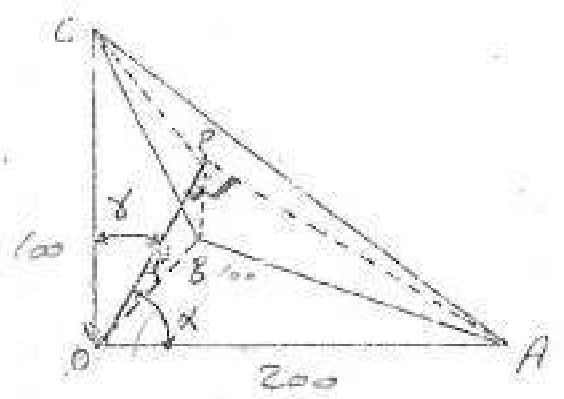
The plane of τ_{max} is 45^0 with the planes of σ_{max} & σ_{min}.

A material fails in tension by $\sigma_t = 80$ MPa & in compression by $\sigma_c = -50$ MPa. The stresses on a rectangular block from this material are: $\sigma_x = 60$ MPa, $\sigma_y = 0$, $\sigma_z = ?$ (to be found), $\tau_{xy} = 40$, $\tau_{yz} = 40$, $\tau_{zx} = 30$, Find σ_z for failure & the direction of the plane of failure?

The stresses on a rectangular block are:
$\sigma_x = 100$ MPa, $\sigma_y = -100$, $\sigma_z = 100$, $\tau_{xy} = \tau_{yx} = 0$, $\tau_{yz} = \tau_{zy} = -50$, $\tau_{zx} = \tau_{xz} = 50$.

Find the normal & shearing stresses on a section cutting the edges to lengths 200 mm, 100 mm & 100 mm in X, Y & Z directions?





$\therefore l^2 + m^2 + n^2 = 1$

$\left(\frac{r}{200}\right)^2 + \left(\frac{r}{100}\right)^2 + \left(\frac{r}{100}\right)^2 = 1$

Construct $\overline{OP} \perp \overline{ABC}$, $\overline{OP} = r$ = length

$l = \cos\alpha = \frac{OP}{OA} = \frac{r}{200}$, $n = \cos\gamma = \frac{OP}{OC} = \frac{r}{100}$

$m = \cos\beta = \frac{OP}{OB} = \frac{r}{100}$ * Find $r = OP$ from ...

Then ; $l = \frac{1}{3}$, $m = \frac{2}{3}$, $n = \frac{2}{3}$

Next find , P_x , P_y & P_z & σ_n. Then find τ_{nt} ; Here $\sigma_n = -11.111\ N/mm^2$; $\tau_{nt} = 129.696\ N/mm^2$.

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الانفعالات

STRAINS

strain is a measure of deformation in a body. ...e, small strains & small deformations are used. Also ...formations & strains in two dimensions in x & y axes are used.

Let:

u be the displacement of a point (x,y) in x-direction.

v be the displacement of a point (x,y) in y-direction.

...here are 3 types of strains in xy-planes:

ϵ_x normal strain in x-direction.

ϵ_y normal strain in y-direction.

γ_{xy} shearing strain in xy-planes.

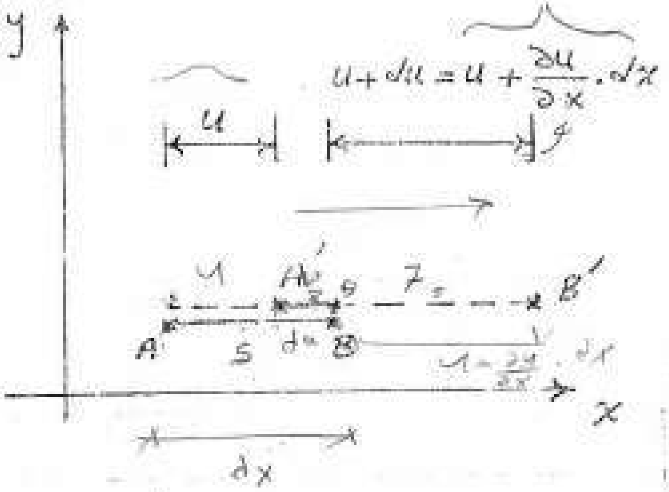
There is a rigid-body rotation ω_z (ω_{xy}) in xy-plane. This does not produce stresses.

Normal strain ϵ_x:

أعتبر

Consider a line AB of ...ength dx (parallel to the x-axis). After movement, ... line is A'B'.

The normal strain ϵ_x is:



$$\epsilon_x = \frac{A'B' - AB}{AB}$$

$$= \frac{\frac{\partial u}{\partial x} \cdot \cancel{dx}}{dx}$$

Thus;

$$\boxed{\epsilon_x = \frac{\partial u}{\partial x}}$$

$\epsilon_x = \frac{\Delta l}{l}$

Normal strain ϵ_y:

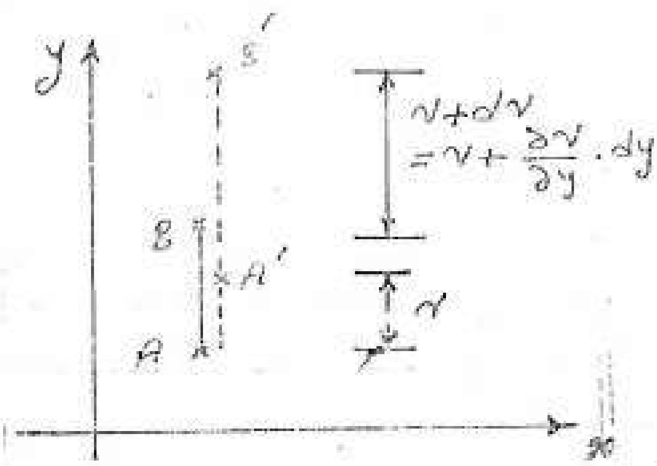
Similarly take a line AB of length dy. Let the line move in y - direction:

Thus the normal strain ϵ_y is:

$$\epsilon_y = \frac{A'B' - AB}{AB}$$

or

$$\boxed{\epsilon_y = \frac{\partial v}{\partial y}}$$



shearing strain γ_{xy} :

Consider two lines AB & AC at right angles ($AB = dx$ & $AC = dy$). When deformation occurs, these two lines rotate into $A'B'$ & $A'C'$.

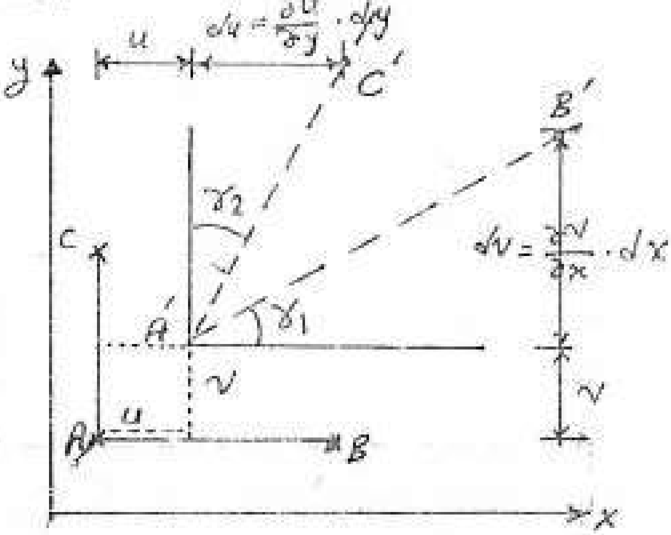
The shearing strain is defined by the sum of two rotations (γ_1 & γ_2). Thus;

$$\boxed{\gamma_{xy} = \gamma_1 + \gamma_2}$$

But, $\gamma_1 = \dfrac{\frac{\partial v}{\partial x} \cdot dx}{dx} = \dfrac{\partial v}{\partial x}$, $\gamma_2 = \dfrac{\frac{\partial u}{\partial y} \cdot dy}{dy} = \dfrac{\partial u}{\partial y}$

Then;

$$\boxed{\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}$$



Rigid-body rotation ω_{xy}: Using the same figure of γ_{xy}, the rigid-body rotation is defined as;

$$\omega_{xy} = \frac{1}{2}(\gamma_2-\gamma_1)$$

Thus; $$\boxed{\omega_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)}$$

Examples:

The displacements in a deformed body (in xy-plane) are given by:

$$u = 0.015\,x^2 y + 0.03$$
$$v = 0.005\,y^2 + 0.03\,xy$$

Find the strains & the rigid-body rotation at a point (0.5, 1.5)?

Solution:

$$\epsilon_x = \frac{\partial u}{\partial x} = 0.030\,xy$$

$$\epsilon_y = \frac{\partial v}{\partial y} = 0.010y + 0.03x$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$
$$= 0.03y + 0.015\,x^2$$

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

$= \frac{1}{2}(0.015x^2 - 0.03y)$

At $(0.5, 1.5)$, put $x = 0.5$ & $y = 1.5$

$\epsilon_x = 0.030 * 0.5 * 1.5 = (0.0225)$

$\epsilon_y = 0.010 * 1.5 * 0.03 * 0.5 = (0.1505)$

$\gamma_{xy} = 0.03 * 1.5 + 0.015 * (0.5)^2 = (\ldots)$ $\frac{\pi}{180} = d$

$\omega_{xy} = \frac{1}{2}\{0.015 * (0.5)^2 - 0.03 * 1.5\}$

$= (-0.020)$ radian.

H.W

1. A line AB with $A(1.0, 0.5)$ & $B(1.4, 0.8)$ has moved to $A'B'$ with $A'(1.1, 0.48)$ & $B'(1.46, 0.88)$. Calculate the strains & the rigid-body rotation in the region of the line AB.

$\in$: Epselon

∂ : partial ١- الانفعال يحسب من الاشتقاق

ω : Omega $\quad \epsilon_x , \epsilon_y , \gamma_{xy}$

٢- قيم u و v يتم حسابها من
خلال التكامل

توافقية الانفعالات

Compatibility of Strains:-

1) If the displacements $u = u(x,y)$ & $v = v(x,y)$ are given, then the strains can be calulated by differentiation:

$$\epsilon_x = \frac{\partial u}{\partial x} , \quad \epsilon_y = \frac{\partial v}{\partial y} , \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

The strain will be compatible.

2) If the strains $\epsilon_x = \epsilon_x(x,y)$, $\epsilon_y = \epsilon_y(x,y)$ & $\gamma_{xy} = \gamma_{xy}(x,y)$ are given, then the displacements u & v must be found by integration. Here, 3-equs are given: $\epsilon_x = \epsilon_x(x,y)$, $\epsilon_y = \epsilon_y(x,y)$, $\gamma_{xy} = \gamma_{xy}(x,y)$.

Only 2 displacement (u & v) are needed. Thus, the strains must be related to each other. This relation is the compatibility equation. To find this relation, come back to, $\epsilon_x = \frac{\partial u}{\partial x}$, $\epsilon_y = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$

Eliminate u & v:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y}\right)$$

$= \gamma_{xy}$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial v}{\partial x}\right)$$

Add:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2}{\partial x \partial y}\Big(\overbrace{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}^{\gamma_{xy}}\Big)$$

$$= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Thus; the compatibility equation is;

$$\boxed{\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}}$$

Examples:

check the following strains,

$$\epsilon_x = m(0.2x^2 - 0.3xy)$$
$$\epsilon_y = m(-0.1xy - 0.2y^2)$$
$$\gamma_{xy} = m(0.4xy - 0.6x^2 + 0.1y^2)$$

where, $m = 10^{-6}$ (a scale factor)

Solution:

use compatibility equation

$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Here:

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = m(0+0) = 0$$

Not Compat.

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = m(0+0) = 0 \quad , \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = m(0.4 - 0 + 0) = 0$$

2) The following strains in xy-plane are given:

$$K\,\epsilon_x = 2\,(x^2+y^2) + x^4 + \frac{1}{2}\,y^4$$

$$K\,\epsilon_y = -4\,(x^2+y^2) + \frac{1}{2}\,x^4 + y^4$$

$$K\,\gamma_{xy} = A\,x\,y\,(x^2+y^2-2)$$

where $K = 10^6$ (a scale factor).
and A is a constant to be determined such that the strains are compatible. Then determine the displacements u & v in x & y-directions. The conditions are $\underline{u(0,0)=0}$, $\underline{v(0,0)=0}$ & $\underline{\omega(0,0)=0}$

Solution: use the compatibility equation:

$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Here:

$$\frac{\partial^2 \epsilon_y K}{\partial x^2} = -8 + 6x^2$$

$$\frac{\partial^2 K\epsilon_x}{\partial y^2} = 4 + 6y^2$$

$$\frac{\partial^2 K\gamma_{xy}}{\partial x \partial y} = A\,(3x^2+3y^2-2)$$

Then, $(-8+6x^2)+(4+6y^2) = A\,(3x^2+3y^2-2)$
This gives, $A = 2$

Then; $k\epsilon_x = 2(x^2+y^2)+x^4+\frac{1}{2}y^4$
$k\epsilon_y = -4(x^2+y^2)+\frac{1}{2}x^4+y^4$
$k\gamma_{xy} = 2(x^3y+xy^3-2xy)$

Start from: $k\epsilon_x = k\dfrac{\partial u}{\partial x}$

Then; $ku = \int k\epsilon_x dx + f(y)$

consider y = constant bcs $\int$ w.r.t (x)

$= \int\{2(x^2+y^2)+x^4+\frac{1}{2}y^4\}dx + f(y)$

$$\boxed{u = \frac{2}{3}x^3+2xy^2+\frac{1}{5}x^5+\frac{1}{2}xy^4+f(y)}$$ C_1

Also, use;
$k\epsilon_y = k\dfrac{\partial v}{\partial y}$ or $kdv = k\epsilon_y\,dy$

Thus $kv = \int k\epsilon_y dy + g(x)$

Conside x: constant bcs $\int$ w.r.t (y)

$= \int\{-4(x^2+y^2)+\frac{1}{2}x^4+y^4\}dy + g(x)$

$$\boxed{V = -4x^2y-\frac{4}{3}y^3+\frac{1}{2}x^4y+\frac{1}{5}y^5+g(x)}$$

from ku & kv find $k\gamma_{xy}$:

$k\gamma_{xy} = \dfrac{k\partial v}{\partial x} + \dfrac{k\partial u}{\partial y}$

$2x$ $\underline{\underline{C}}x$ $g(x)$ Cy

2 & $F(y)$ Cx

$(x, y) = 0$ $\left(\frac{dg}{dx}\right)$ $C(x)$ 25

$$k\gamma_{xy} = \frac{k\partial v}{\partial x} + \frac{k\partial u}{\partial y}$$

$$= \left\{-8xy + 0 + 2x^3y + 0 + \overset{c_1}{\frac{dg}{dx}}\right\}$$

$$+ \left\{0 + 4xy + 0 + 2xy^3 + \overset{c_2}{\frac{df}{dy}}\right\}$$

$\frac{dg}{dx}$ or $\underline{\frac{df}{dy}}$ → function for $\underline{\underline{1}}$ variable (x or y), ordinary ; not partial ?

هي دالة لمتغير واحد فقط ، أي اعتيادية وليست جزئية

put this equal to the given $\underline{k\gamma_{xy}}$. Thus

$$-4xy + 2x^3y + \frac{dg}{dx} + 2xy^3 + \frac{df}{dy} = 2(x^3y + xy^3 - 2xy)$$

Thus; $\frac{dg}{dx} + \frac{df}{dy} = 0$ or $\underbrace{\frac{dg}{dx}}_{\text{function of } x} = -\underbrace{\frac{df}{dy}}_{\text{function of } y}$

$c_1 = c_2$

ملاحظة : من غير الممكن أن تتساوى دالتين لمتغيرين مختلفين إلا عندما تكونا ثوابت (constants) . ! ?

Thus is only possible if $\frac{dg}{dx} = C$ (constant)

$\frac{df}{dy} = C$ (same constant)

Take;

$$\frac{dg}{dx} = c \quad \text{or} \quad g = cx + A_1$$
$$\frac{df}{dy} = -c \quad \text{or} \quad f = -cy + A_2$$

by $\int$?

Then;

$$Ku = \frac{2}{3}x^3 + 2xy^2 + \frac{1}{5}x^5 + \frac{1}{2}xy^4 + [f(y) = -cy + A_2]$$

$$= \frac{2}{3}x^3 + 2xy^2 + \frac{1}{5}x^5 + \frac{1}{2}xy^4 - cy + A_2$$

$$Kv = -4x^2y - \frac{4}{3}y^3 + \frac{1}{2}x^4y + \frac{1}{5}y^5 + cx + A_1$$

Find A_1, A_2 & C from the given conditions. Thus

$A_2 = 0$, $A_1 = 0$ } when x & $y = 0 \Rightarrow A_1, A_2 = 0$, substit. in equs.

Find C from $\quad \omega = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$

or $\quad 2K\omega = \frac{K\partial u}{\partial y} - \frac{K\partial v}{\partial x}$

$$= (4xy + 2xy^3 - c) - (-8xy + 2x^3y + c)$$

put $\omega = 0$ at $x = y = 0$. Then $C = 0$

(where x & $y = 0 \rightarrow$ the body remain without rotate?)

Then the displacements are:

$$Ku = \frac{2}{3}x^3 + 2xy^2 + \frac{1}{5}x^5 + \frac{1}{2}xy^4$$

$$Kv = -4x^2y - \frac{4}{3}y^3 + \frac{1}{2}x^4y + \frac{1}{5}y^5$$

~~2) check ; come back to main compatible equ & change~~

Q The strain in xy plan are . Find the desplesment you equall x, y and $V=V(x,y)$ in x-y directions respect the given condition's are $u(0,0)=0$, $V(0,0)=0$ $w(0,y)=0$?

H.W :-

$$\left.\begin{aligned} \epsilon_x &= 5 + x^2 + y^2 + x^4 + y^4 \\ \epsilon_y &= 6 + 3x^2 + 3y^2 + x^4 + y^4 \\ \gamma_{xy} &= 10 + 4x^3y + 4xy^3 + 8xy \end{aligned}\right\}$$

$12y^2$

Solution :- cheack the combatibility

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y}$$

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = 0 + 0 + 2 + 0 + 4*3y^2 \Rightarrow \frac{\partial^2 \epsilon_x}{\partial y^2} = 2 + 12y^2$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = 0 + 6 + 0 + 12x^2 + 0 \Rightarrow \frac{\partial^2 \epsilon_y}{\partial x^2} = 6 + 12x^2$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y} = 0 + 4*3x^2 + 12y^2 + 8 \Rightarrow \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 12x^2 + 12y^2 + 8$$

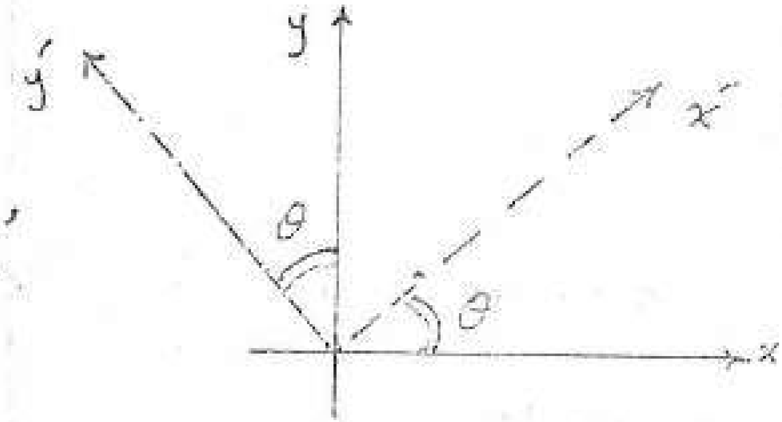
[illegible] =

when sub 0 ⇒ 8 = 8 ∴ Compatiblity

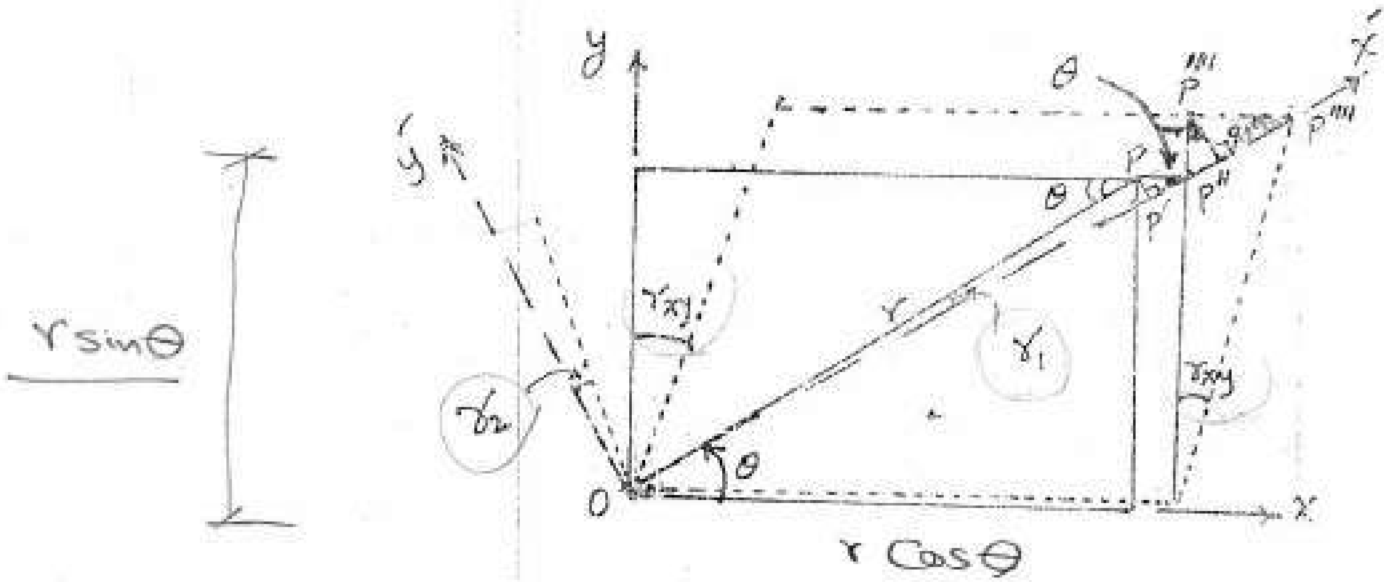
عبدالرحمن عوبد الله حمد

Plane strains In Oblique directions:

Suppose that ϵ_x, ϵ_y & γ_{xy} are given along x & y- directions. It is required to find the strain $\epsilon_{x'}, \epsilon_{y'}$ & $\gamma_{x'y'}$ along the oblique directions x' & y'.



To derive the formula, consider a rectangular element under strains ϵ_x, ϵ_y & γ_{xy}.



Here:

$$\epsilon_x = \frac{PP''}{r\cos\theta} \rightarrow PP'' = \epsilon_x \cdot r\cos\theta$$

$$\epsilon_y = \frac{P''P'''}{r\sin\theta} \rightarrow P''P''' = \epsilon_y \cdot r\sin\theta$$

$$\gamma_{xy} = \frac{P'''P''''}{r\sin\theta} \rightarrow P'''P'''' = \gamma_{xy} \cdot r\sin\theta$$

Derive $\epsilon_{x'}$ (in (r) or (x') direction).
The extension of diagonal "r" is $P'P''''$, then:

$$\epsilon_{x'} = \frac{P'P''''}{r}$$

Calculate the extension $P'P''''$ Here:

$$P'P'''' = P'P'' + P''P''''$$

or:

$$P'P'''' = PP'' \cdot \cos\theta + P''P''' \cdot \sin\theta + P'''P'''' \cdot \cos\theta$$

$$= (\epsilon_x \cdot r\cos\theta) \cdot \cos\theta + (\epsilon_y \cdot r\sin\theta) \cdot \sin\theta + (\gamma_{xy} \cdot r\sin\theta) \cdot \cos\theta$$

Thus, $\epsilon_{x'} = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cdot \cos\theta$

For $\epsilon_{y'}$, use $(\theta + \frac{\pi}{2})$ in the formula for $\epsilon_{x'}$. Thus:

$$\epsilon_{y'} = \epsilon_x \cos^2(\theta + \tfrac{\pi}{2}) + \epsilon_y \cdot \sin^2(\theta + \tfrac{\pi}{2}) + \gamma_{xy} \sin(\theta + \tfrac{\pi}{2}) \cdot \cos(\theta + \tfrac{\pi}{2})$$

or,

$$\epsilon_{y'} = \epsilon_x \sin^2\theta + \epsilon_y \cos^2\theta - \gamma_{xy} \sin\theta \cos\theta$$

For $\gamma_{x'y'}$, use $\gamma_1 = \frac{PP'}{r} = \frac{PP''\sin\theta}{r} = \frac{\epsilon_x r\cos\theta\sin\theta}{r}$
$= \epsilon_x \cos\theta \sin\theta$

Also, $\gamma_2 = \epsilon_y \sin\theta \cos\theta$

Then $\gamma_1 + \gamma_2 = \gamma_{x'y'} = (\epsilon_x + \epsilon_y)\sin\theta\cos\theta$

Rewriting;

$$\epsilon_{x'} = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$
$$\epsilon_{y'} = \epsilon_x \sin^2\theta + \epsilon_y \cos^2\theta - \gamma_{xy}\sin\theta\cos\theta$$

$$\frac{1}{2}\gamma_{x'y'} = -\frac{(\epsilon_x - \epsilon_y)}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Use, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

Then;

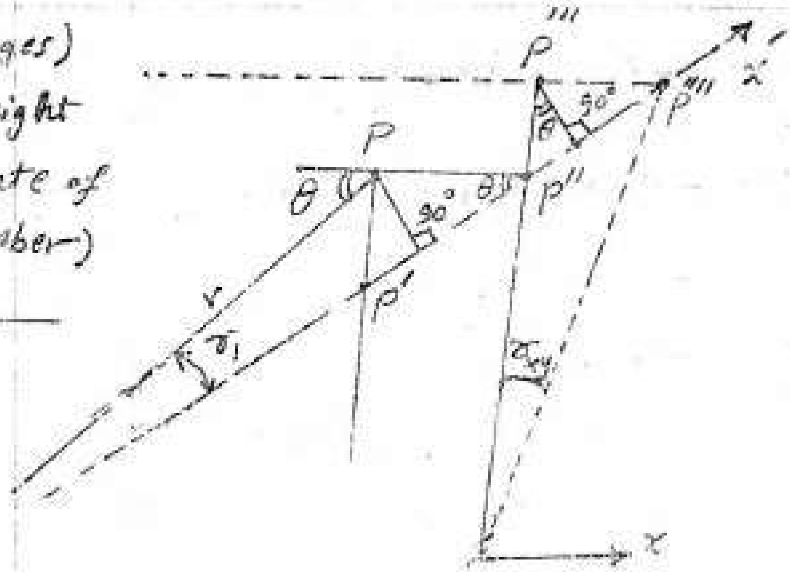
$$\epsilon_{x'} = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\epsilon_{y'} = \frac{1}{2}(\epsilon_x + \epsilon_y) - \frac{1}{2}(\epsilon_x - \epsilon_y)\cos 2\theta - \frac{1}{2}\gamma_{xy}\sin 2\theta$$

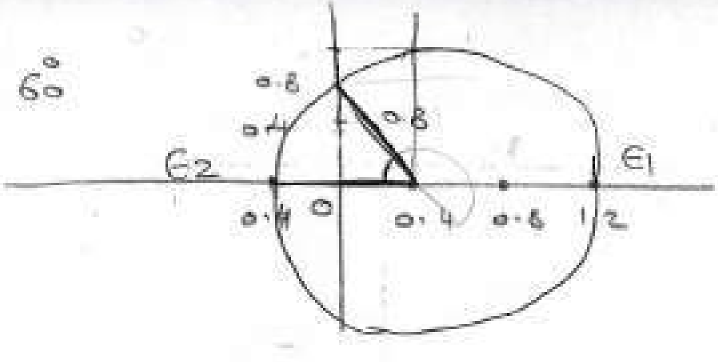
$$\frac{\gamma_{x'y'}}{2} = -\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

or
$$\frac{\gamma_{x'y'}}{2} = -\frac{(\epsilon_x - \epsilon_y)}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

* Finite strains (or changes)
. let the arc as a straight distance, $\frac{\text{arc}}{\text{radius}}$ = (rotate of member)



$\cos\phi = \frac{0.4}{0.8} = 60^\circ$



Principal Strains:-

These occur in directions of zero shearing strains. One principal strain is maximum normal strain ϵ_1 & the other principal strain is minimum normal strain ϵ_2. To get ϵ_1 or ϵ_2 take ϵ_x' & use $\frac{\partial \epsilon_x'}{\partial \theta} = 0$

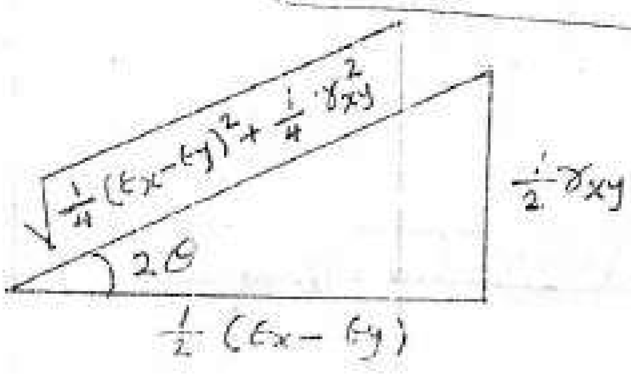
Then
$$\epsilon_x' = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y)\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

Here
$$\frac{\partial \epsilon_x'}{\partial \theta} = 0 + \frac{1}{2}(\epsilon_x - \epsilon_y)(-2\sin 2\theta) + \frac{1}{2}\gamma_{xy} \cdot (2\cos 2\theta) = 0$$

This gives,
$$\tan 2\theta = \frac{\frac{1}{2}\gamma_{xy}}{\frac{1}{2}(\epsilon_x - \epsilon_y)}$$

substitute, then;
$$\epsilon_1 = \frac{1}{2}(\epsilon_x + \epsilon_y) + \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

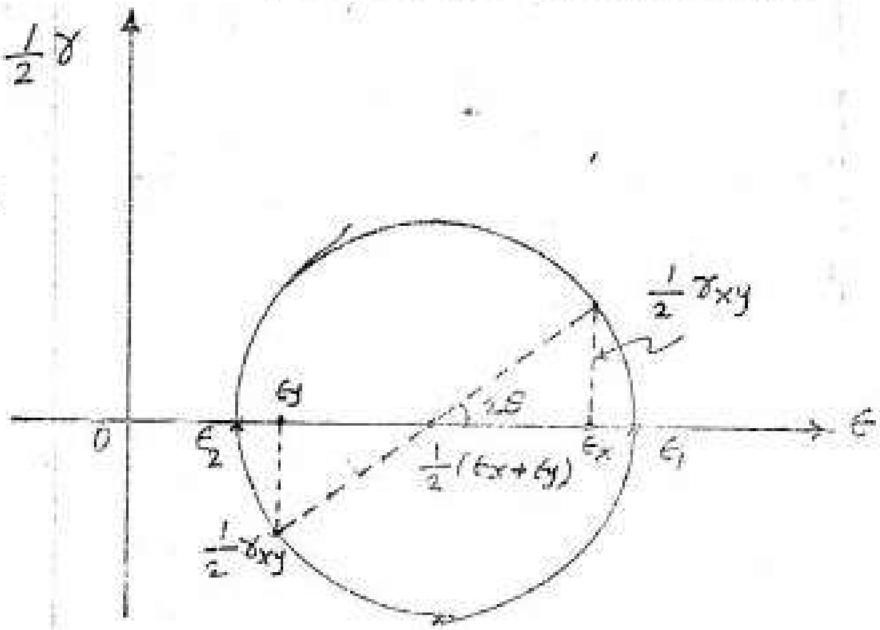
Also;
$$\epsilon_2 = \frac{1}{2}(\epsilon_x + \epsilon_y) - \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$



These can be obtained from <u>Mohr circle of strains.</u>

(shear strain = $\frac{1}{2}$ angle of rotation)

check with Mohr's circle for stresses?



Problems of 2-D strains:

1) If the principal strains ϵ_1 & ϵ_2 are given, then it is possible to find the strains ϵ_x', ϵ_y' & $\gamma_{x'y'}$ in any direction θ. By Mohr's circle of strains or use the formula.

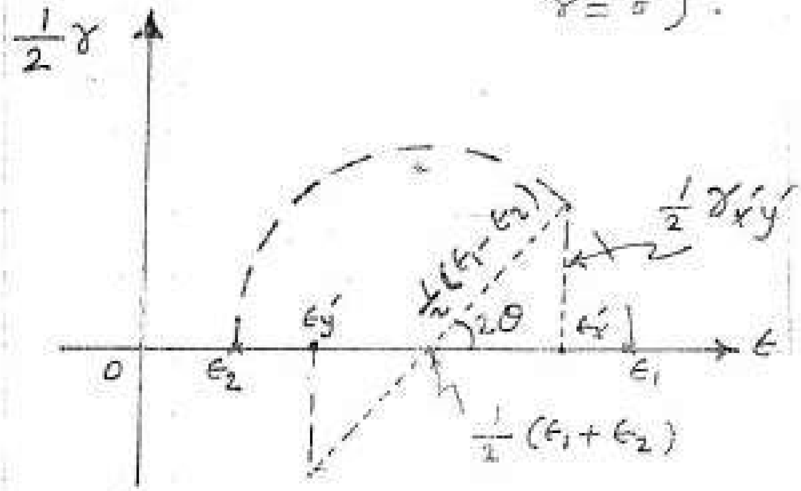
$$\epsilon_x' = \epsilon_1 \cos^2\theta + \epsilon_2 \sin^2\theta$$
$$\epsilon_y' = \epsilon_1 \sin^2\theta + \epsilon_2 \cos^2\theta$$

or

$$\epsilon_x' = \frac{1}{2}(\epsilon_1+\epsilon_2) + \frac{1}{2}(\epsilon_1-\epsilon_2)\cos 2\theta$$

$$\epsilon_y' = \frac{1}{2}(\epsilon_1+\epsilon_2) - \frac{1}{2}(\epsilon_1-\epsilon_2)\cos 2\theta$$
$$\frac{1}{2}\gamma_{x'y'} = -\frac{1}{2}(\epsilon_1-\epsilon_2)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

where here, $\epsilon_x = \epsilon_1$, $\epsilon_y = \epsilon_2$ & $\gamma_{xy} = 0$ (principal strains $\gamma = 0$).

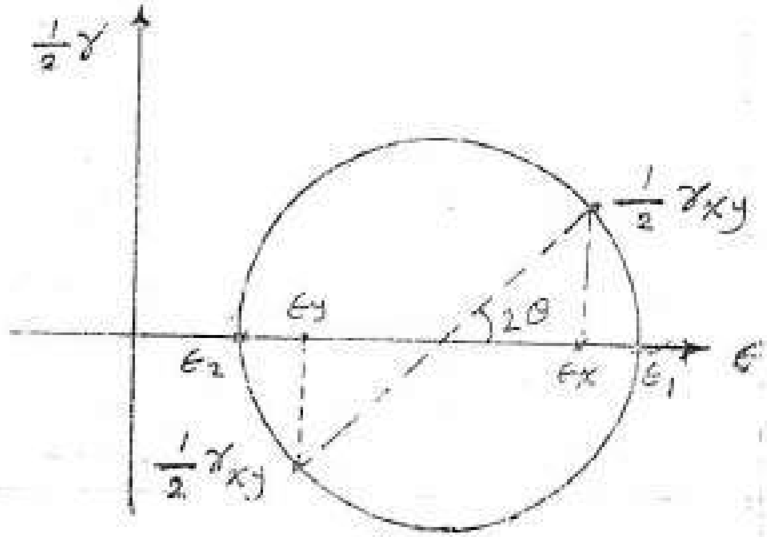


2) If ϵ_x, ϵ_y & γ_{xy} are given, then the principal strains ϵ_1 & ϵ_2 can be obtained.

$$\epsilon_{1,2} = \frac{1}{2}(\epsilon_x + \epsilon_y) \mp \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

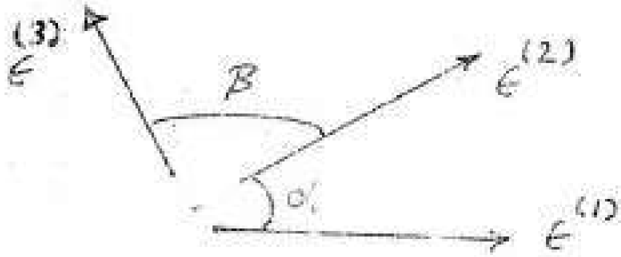
$$\tan 2\theta = \frac{\frac{1}{2}\gamma_{xy}}{\frac{1}{2}(\epsilon_x - \epsilon_y)} \quad \text{(direction)}$$

Here, $\epsilon_x' = \epsilon_1$
$\epsilon_y' = \epsilon_2$
$\gamma_{x'y'} = 0$



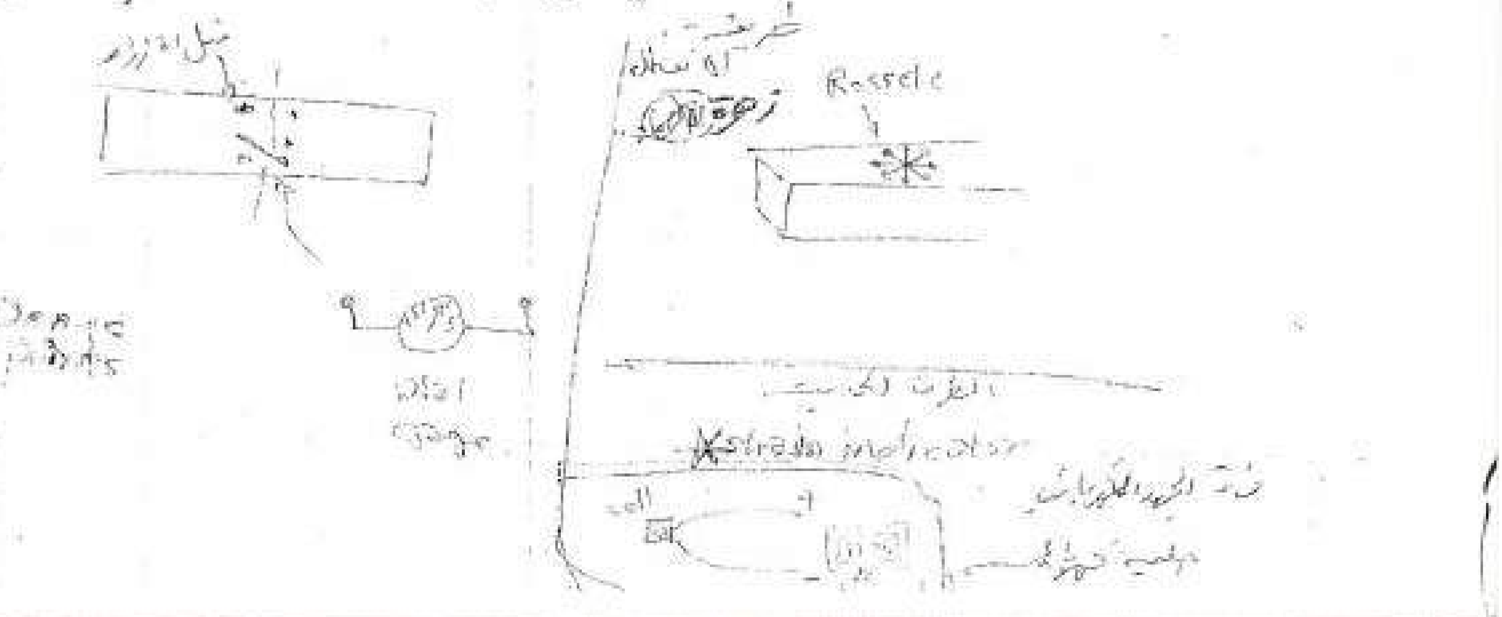
Problems in the measurement of strains in two directions:

Usually normal strains can be measured easily but shearing strains cannot be measured. For this reason, 3 normal strains must be measured in order to obtain principal strains ϵ_1 & ϵ_2 & the directions "θ".



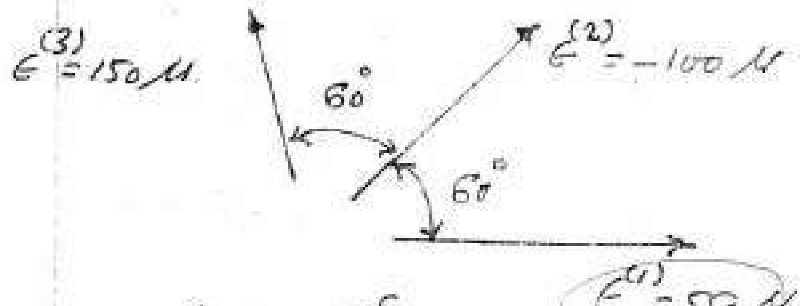
First find ϵ_x, ϵ_y & γ_{xy}.
Let $\epsilon_x = \epsilon^{(1)}$ by taking x-axis along $\epsilon^{(1)}$ then find ϵ_y & γ_{xy} by using $\epsilon^{(2)} = \epsilon_x'$, at $\theta = \alpha$ & repeat $\epsilon^{(3)} = \epsilon_x'$ at $\theta = \alpha + \beta$.

When ϵ_x, ϵ_y & γ_{xy} are found, then find the principal strains ϵ_1 & ϵ_2.



Examples: (2 - D strains):

) Three normal strains in three directions in a plate in plane stresses are measured as shown in the figure. Find the principal strains ϵ_1, ϵ_2 & their directions. use $E = 200$ GPa & $\nu = 0.3$ to find the principal stresses.



($\mu = 10^{-6}$ or micro)

Solution: choose x-axis along $\epsilon^{(1)}$, Then $\epsilon_x = \epsilon^{(1)}$
$= 550\mu$

Next find ϵ_y & γ_{xy}. use $\epsilon^{(2)} = \epsilon_x'$, then (with $\theta = 60°$);

$$\underbrace{-100\mu}_{\epsilon_x'} = \epsilon_x \cos^2\theta + \epsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta$$

$$-100\mu = 550\mu * \frac{1}{4} + \epsilon_y * \frac{3}{4} + \gamma_{xy} * \frac{\sqrt{3}}{2} * \frac{1}{2}$$

$$= \frac{550\mu}{4} + \frac{3}{4}\epsilon_y + \frac{\sqrt{3}}{4}\gamma_{xy} \quad \text{.....} \; (1)$$

Repeat by using: $\epsilon^{(3)} = \epsilon_x'$ with $\theta = 120°$

$\theta = \alpha + \beta$

Then ; $150\mu = 550\mu \cos^2(\theta = 120^\circ) + \epsilon_y \sin^2 120^\circ$
$+ \gamma_{xy} \sin 120^\circ \cos 120^\circ$

or

$$150\mu = 550\mu * \left(\frac{-1}{2}\right)^2 + \epsilon_y * \left(\frac{\sqrt{3}}{2}\right)^2$$

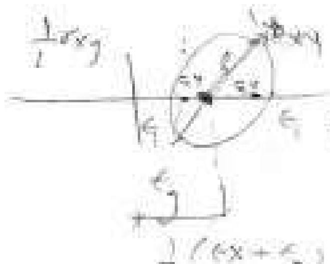
$$+ \gamma_{xy} * \left(\frac{\sqrt{3}}{2}\right) * \left(-\frac{1}{2}\right)$$

$$150\mu = \frac{550\mu}{4} + \frac{3}{4}\epsilon_y - \frac{\sqrt{3}}{4}\gamma_{xy} \quad \ldots\ldots (2)$$

Solve : $\epsilon_y = -150\mu$, $\gamma_{xy} = -\frac{500}{\sqrt{3}}\mu$

Then, the principal strains are , $\epsilon_{1,2} = \frac{1}{2}(\epsilon_x + \epsilon_y)$

$$\mp \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$



Substitute, $\epsilon_1 = 579\mu$ (tensile)
$\epsilon_2 = -179\mu$ (Comp.)

For the principal stresses, use ;

$$\sigma_1 = \frac{E}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2)$$

$$\sigma_2 = \frac{E}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1)$$

substitute, then ; $\sigma_1 = 115\ MPa$ (tens.)
$\sigma_2 = -1.2\ MPa$ (Comp.)

The directions of principal strains or stresses

$$\tan 2\theta = \frac{\frac{1}{2}\gamma_{xy}}{\frac{1}{2}(\epsilon_x - \epsilon_y)}$$; substitute, then $\tan 2\theta = -0.4125$

This gives : $2\theta = -22.4^\circ$ & $180 - 22.4^\circ$

Thus ; $\theta = \underbrace{-11.2^\circ \ \& \ 78.8^\circ}$
(at right angles).

$G = \frac{\tau}{\gamma}$, $G =$

THEORIES OF FAILURE

Failure in material is a state where the material cannot take any more load (or stress).

Failure in structures may be انبعاج buckling or by فشل yielding or تهشيم crushing. Here, only yielding or crushing will be considered.

Failure of a material under one normal stress is assumed to be known from testing. It is required to find a formula to predict تنبأ / يتكهن failure.

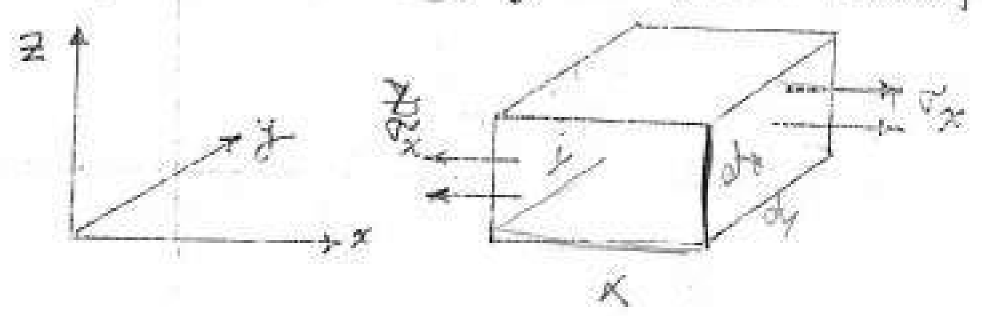
When the material is under all normal & shearing stresses. Reference to the failure under the tensile or compressive stress will be made.

بسبب طاقة الانفعال

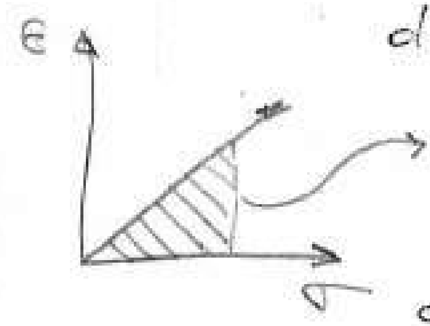
Strain Energy Due to stresses:

1. A material under one normal stress σ_x.

Consider a rectangular block $dx\,dy\,dz$ under normal stress σ_x.



The strain in x-direction is ϵ_x. The extension will be $\epsilon_x . dx$. The work is equal to the strain energy.



$$dU = dW \quad \overset{P}{\sigma * A}$$

$$= \frac{1}{2} \underbrace{(\sigma_x . dy . dz)}_{force} \; \underbrace{(\epsilon_x . dx)}_{movement}$$

$$dU = \frac{1}{2} \epsilon_x \sigma_x (dx\, dy\, dz)$$

$$= \frac{1}{2} \epsilon_x \sigma_x \, d(volume)$$

$(\epsilon_x = \frac{\Delta}{(dx)} \rightarrow \Delta = dx . \epsilon_x)$

The strain energy (per unit volume) $\frac{dU}{d(vol.)}$ will be:

$$U' = \frac{1}{2} \epsilon_x \sigma_x$$

2.. strain energy due to another normal stress σ_y, Similarly, the strain energy per unit volume is,

$$U' = \frac{1}{2} \epsilon_y \sigma_y$$

strain energy due to normal stress σ_z, Also, the strain energy per unit volume is, $U' = \frac{1}{2} \epsilon_z \sigma_z$

THEORIES OF FAILURE

Failure in material is a state where the material cannot take any more load (or stress).

Failure in structures may be buckling or by yielding or crushing. Here, only yielding or crushing will be considered.

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Strain Energy Due to stresses:

A material under one normal stress σ_x.

Consider a rectangular block $dx\,dy\,dz$ under normal stress σ_x.

The strain in x-direction is ϵ_x. The extension will be $\epsilon_x . dx$. The work is equal to the strain energy.

$$dU = dW$$

$$= \frac{1}{2} \underbrace{(\sigma_x . dy . dz)}_{force} . \underbrace{(\epsilon_x . dx)}_{movement}$$

$$dU = \frac{1}{2} \epsilon_x \sigma_x (dx\,dy\,dz)$$

$$= \frac{1}{2} \epsilon_x \sigma_x \, d(volume)$$

$\left(\epsilon_x = \frac{\Delta}{(dx)} \Rightarrow \Delta = dx . \epsilon_x\right)$

The strain energy (per unit volume) $\frac{dU}{d(Vol.)}$ will be :

$$\boxed{U' = \frac{1}{2} \epsilon_x \sigma_x}$$

strain energy due to another normal stress σ_y, Similarly, the strain energy per unit volume is,

$$\boxed{U' = \frac{1}{2} \epsilon_y \sigma_y}$$

strain energy due to normal stress σ_z, Also, the strain energy per unit volume is, $\boxed{U' = \frac{1}{2} \epsilon_z \sigma_z}$

4. strain energy due to one shearing stress $\underline{\tau_{xy} = \tau_{yx}}$,
Consider the block $dx\,dy\,dz$ under four shearing stresses τ_{xy} & τ_{yx}.

$\underset{dW}{\underline{\text{The work}}}$ is equal to the $\underset{dU}{\underline{\text{strain energy}}}$. Thus;

$$dW = dU$$

or $dW = dU$

$$= \frac{1}{2} \underbrace{(\tau_{yx} \cdot dx \cdot dz)}_{\text{force}} \underbrace{(\gamma_{xy} \cdot dy)}_{\text{movement}}$$

Then, the strain energy per unit volume is;

$$\boxed{U' = \frac{dU}{d(\text{vol.})} = \frac{1}{2}\, \gamma_{xy}\, \tau_{xy}} \quad ; \underline{\tau_{xy} = \tau_{yx}}$$

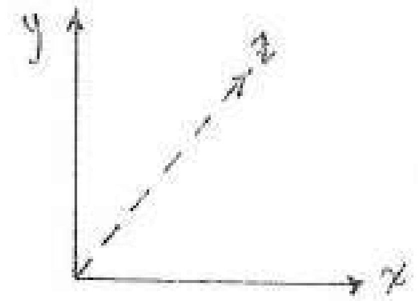
5. Strain energy due to another shearing stress $\tau_{yz} = \underline{\tau_{zy}}$
similarly, the strain energy per unit volume is;

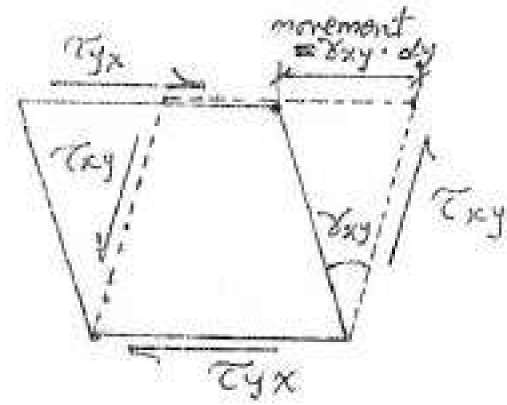
$$\boxed{U' = \frac{1}{2}\, \gamma_{yz}\, \tau_{yz}}$$

6. strain energy due to the shearing stress $\tau_{xz} = \underline{\tau_{zx}}$,
Also, the strain energy per unit volume;

$$\boxed{U' = \frac{1}{2}\, \gamma_{zx}\, \tau_{zx}}$$

[illegible] γ_{yx} γ_{zy} γ_{zx}





Total strain energy per unit volume (or energy density):
Add the strain energy per unit volume from all normal & shearing stresses, Then;

$$U' = \frac{1}{2}\epsilon_x\sigma_x + \frac{1}{2}\epsilon_y\sigma_y + \frac{1}{2}\epsilon_z\sigma_z + \frac{1}{2}\gamma_{xy}\tau_{xy} + \frac{1}{2}\gamma_{yz}\tau_{yz} + \frac{1}{2}\gamma_{zx}\tau_{zx}$$

Use,

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$\gamma_{xy} = \frac{\tau_{xy}}{G}$, $\gamma_{yz} = \frac{\tau_{yz}}{G}$, $\gamma_{zx} = \frac{\tau_{zx}}{G}$; G: modulus of rigidity or shearing modulus of Elasticity

and substitute, then;

$$U' = \frac{1}{2E}\left\{(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)\right\} + \frac{1}{2G}\left\{(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right\}$$

check!!

When only principal stresses (σ_1, σ_2 & σ_3) are acting,

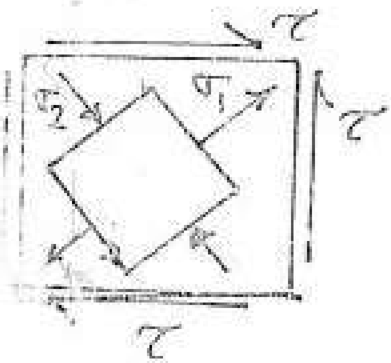
$$U = \frac{1}{2E}\left\{(\sigma_1^2+\sigma_2^2+\sigma_3^2) - 2\nu\,(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right\}$$

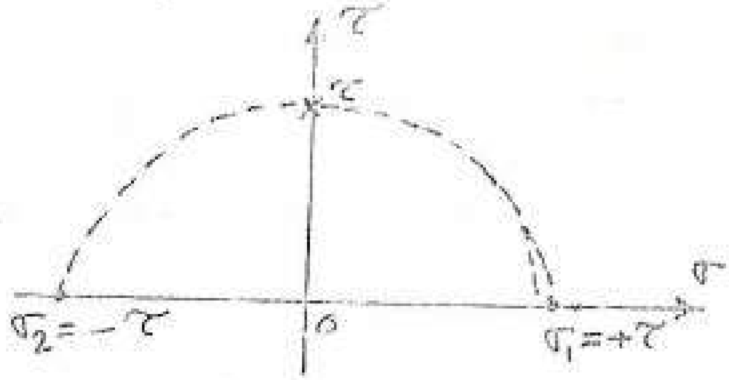
Here $\boxed{\tau = 0}$ → because principal stresses.

Remember, $\boxed{G = \frac{E}{2(1+\nu)}}$, In order to proof of

$G = \frac{E}{2(1+\nu)}$?

Consider a square plate ($a \times a$) under shearing stresses τ on all four edges. This plate will have a tensile stress $\sigma_1 = +\tau$ along one diagonal & a compressive stress $\sigma_2 = -\tau$ along the other diagonal.



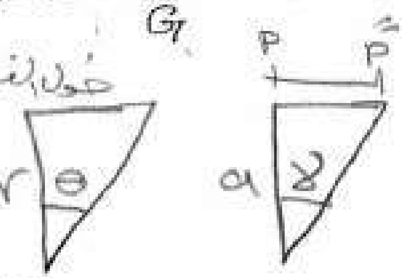


Consider the deformation of this plate.

Here $pp'' = a.\gamma$

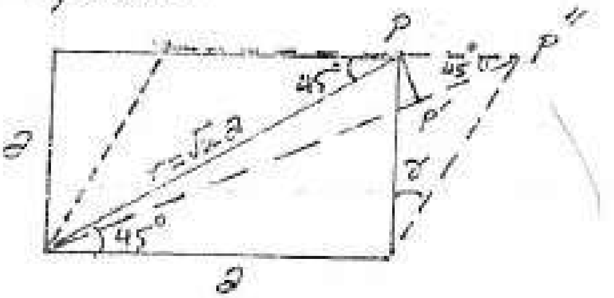
$= a.\frac{\tau}{G}$

[illegible]



$PP'' = a\gamma$

[illegible] $= r \times \theta$



The strain ϵ_r along the diagonal is:

$$\epsilon_r = \frac{P'P''}{r} \quad \text{(as } P'P'' \text{ is extension in length } r\text{)}$$

$$\epsilon_r = \frac{PP'' \cos 45^\circ}{\sqrt{2}\, a} = \frac{a\tau}{G} \cdot \frac{1}{2a} = \frac{\tau}{2G}$$

But;

$$\epsilon_r = \frac{\sigma_1}{E} - \nu \frac{\sigma_2^*}{E}$$

* ; σ_2 perpendicular dir.

$$= \frac{\tau}{E} - \nu \frac{-\tau}{E} = \frac{\tau}{E}(1+\nu)$$

Then, $\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$; so $\boxed{G = \frac{E}{2(1+\nu)}}$

اشتقاق . مهم .

مهم جداً هذا الاشتقاق

الضغط الهايدروستاتيكي
الساكن

تركيب مادة

* Decomposition of Stresses into hydrostatic & deviatoric stresses:

Consider a rectangular block under principal stresses σ_1, σ_2 & σ_3. These stresses can be decomposed into hydrostatic stresses.

$$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \qquad \text{(hydrostatic stresses)}$$

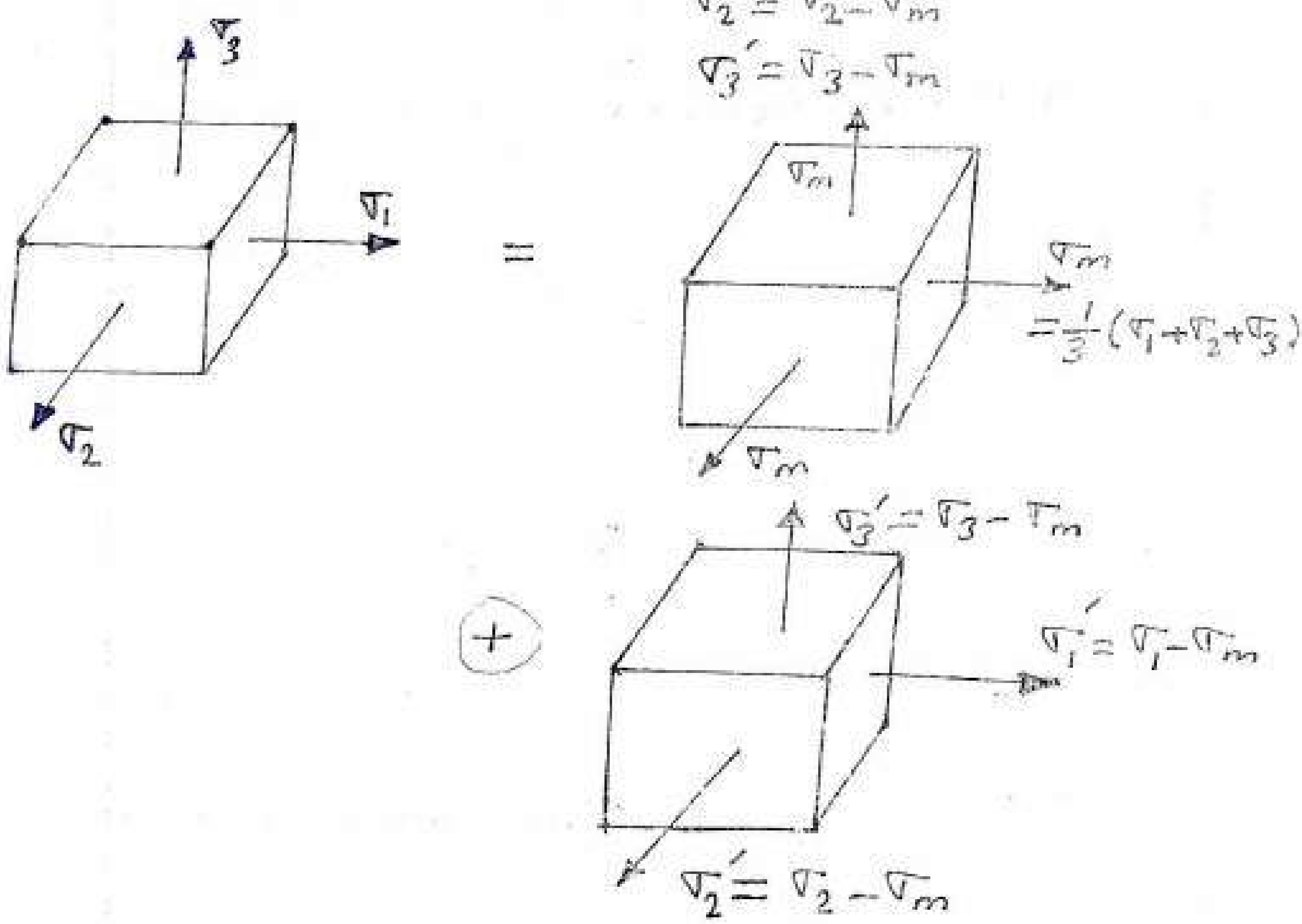
معدل الإجهادات   الإجهاد العرضي

$$\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

and into deviatoric stresses:

الإجهاد العرضي.

$$\sigma_1' = \sigma_1 - \sigma_m$$
$$\sigma_2' = \sigma_2 - \sigma_m$$
$$\sigma_3' = \sigma_3 - \sigma_m$$



Hydrostatic stress $\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ produce a change in volume (volumetric strain) & no shearing strain. The strain energy per unit volume due to hydrostatic stresses is:

$$U' = \frac{1}{2E}\left\{(\sigma_m^2 + \sigma_m^2 + \sigma_m^2) - 2\nu(\sigma_m \cdot \sigma_m + \sigma_m \cdot \sigma_m + \sigma_m \cdot \sigma_m)\right\}$$

* إذا تعرض الجسم إلى ضغط من جميع الجهات
فانه يعني ينضغط ؟ وليس فيه انفعال قص؟

$$U' = \frac{1}{2E}\left\{3\sigma_m^2 - 6\nu\sigma_m^2\right\}$$

$$= \frac{3\sigma_m^2}{2E}(1-2\nu)$$

The deviatoric stresses produce only shearing strains (& no change in volume). The strain energy per unit volume is:

$$U' = \frac{1}{2E}\left\{(\sigma_1'^2 + \sigma_2'^2 + \sigma_3'^2) - 2\nu(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_1')\right\}$$

$$= \frac{1+\nu}{3E}\left\{(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right\}$$

$$= \frac{1}{12G}\left\{\underline{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}\right\}$$

H.W. Check σ??

Theories of failure:

There are various theories;

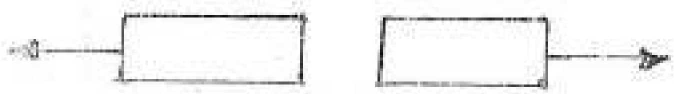
1. The theory of maximum principal stress (Rankine Theory):

Failure occurs when the maximum principal stress reaches a certain value K. Let σ_1 be the maximum principal stress. Then failure occurs when:

$\sigma_1 = K$) ; For material under a tensile (or compressive) stress, failure occurs when $f_y = K$ (yield stress). Thus the formula is:

$$\boxed{\sigma_1 = f_y}$$

this theory is good for brittle materials (such as concrete, chalk, ...).



(flat failure surfaces)

In this theory, the other two principal stresses have no effect.

The theory of maximum shearing stress (Tresca theory):

Here, failure occurs when the maximum shearing stress reaches a certain value K. Let $\sigma_1 > \sigma_2 > \sigma_3$ be the principal stresses, then;

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

(radius of Mohr circle)

* إذا بلغ إجهاد القص حد معين تفشل المادة

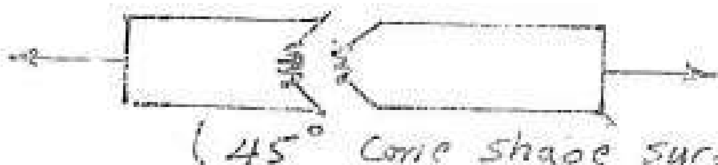
For a material under one tensile (or compressive) stress (fy) at failure, then write $\sigma_1 = fy$ & $\sigma_3 = 0$, thus;

$$\tau_{max} = \frac{1}{2}(fy - 0) = \frac{fy}{2}$$

But $\tau_{max} = K$ for both cases, then failure occurs when;

$$\frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2} fy \quad \text{or} \quad \boxed{\sigma_1 - \sigma_3 = fy}$$

Here, the intermediate stress σ_2 has no effect. This theory is good for ductile materials (such as steel, ...).



(45° cone shape surface)

* لاحظ القطع هنا سطح الفشل غير مستوي (flat)

3. The theory of maximum distortion (or shear) energy (Von Mises theory):

This is a very good theory. Here, failure occurs when the shear (or distortion) energy reaches a certain value (K).

Let σ_1, σ_2 & σ_3 be the principal stresses in a material at failure. The strain energy per unit volume due to shear is:

$$U' = \frac{1}{12G} \left\{ (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 \right\}$$

When the material fails in one tensile (or compressive) stress, then write $\sigma_1 = fy$, $\sigma_2 = 0$ & $\sigma_3 = 0$. Thus,

$$U' = \frac{1}{\cancel{12}\,G} \left\{ (fy-0)^2 + (0-0)^2 + (0-fy)^2 \right\} = \frac{fy^2}{6G}$$

Equating both to K, then failure occurs when;

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2fy^2$$

(This is a very good theory for ductile materials).

* نظرية مهمة للمواد المطيلية (القابلة للسحب) مثل الحديد؟

$$U = \frac{1}{2G} fy^2$$

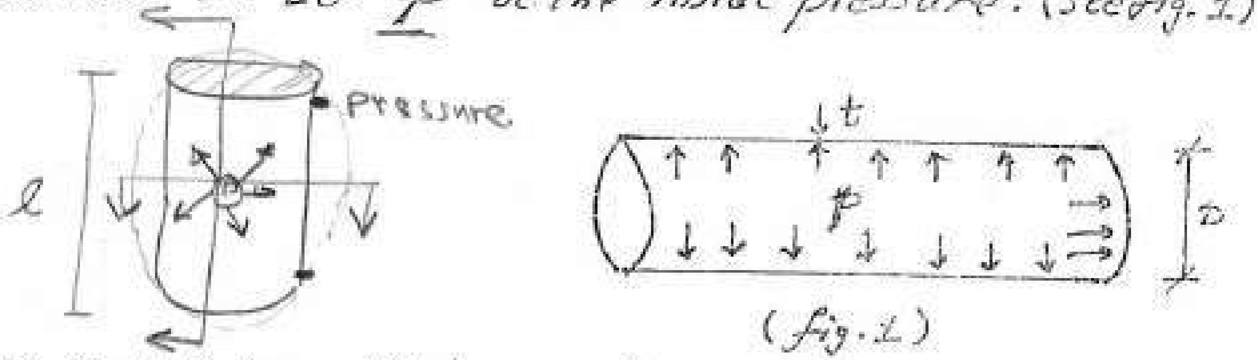
الأوعية vessels shape Cylindrecal

نظريات الفشل تطبيقات

Applications of the Theories of failure:-

1. Circular Cylinderical vessels Under pressure:

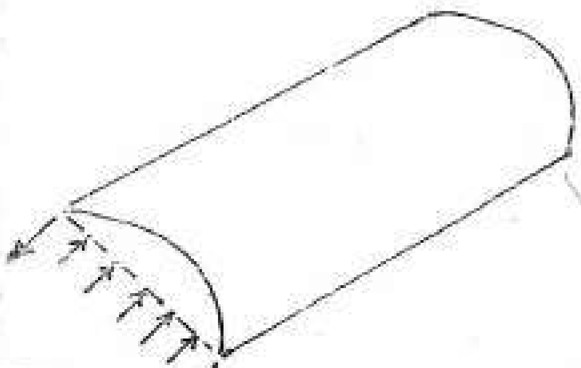
Consider a cylindrical vessel of mean diameter D & small thickness t. Let p be the inside pressure (الضغط الداخلي). (see fig. 1).

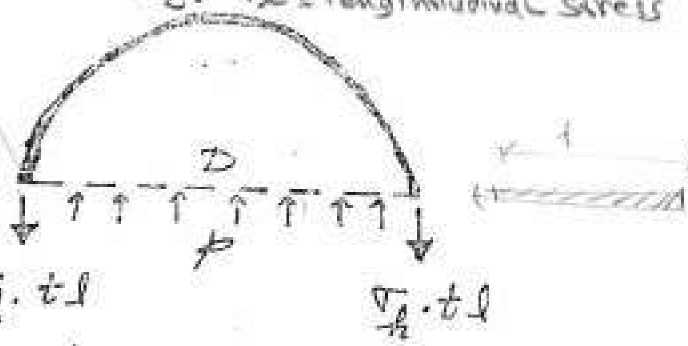


(fig. 1)

This cylinder will have a hoop normal stress σ_h & longitudinal normal stress σ_l. To find σ_h. take cylinder. (see fig. 2). Assume the fluid inside $\frac{1}{2}$ cylinder as a connected part to the cylinder.

الإجهاد الكلي
1. σ_h = hoop stress
2. σ_l = longitudinal stress





$\boxed{\sigma_h = 2\sigma_l}$

ملاحظة مهمة

(fig. 2)

* الضغط العالي في داخل الاسطوانة الذي يسلط المائع يسبب إجهادات!

Use equilibrium of forces:

$$2\sigma_h \cdot \underbrace{tl}_{\text{area of wall}} = p \cdot \underbrace{Dl}_{\text{area of pressure}}$$

$$\sigma_h = \frac{PD}{2t}$$

Thus;

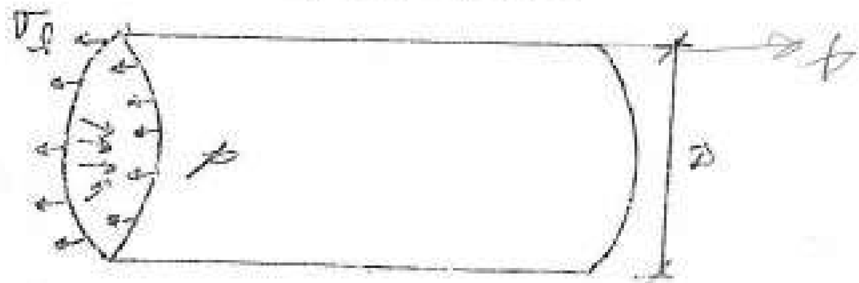
$$\boxed{\sigma_h = \frac{pD}{2t}}$$

To find σ_l take a x-section, Here;

$$\sigma_l \cdot \underbrace{(\pi D \cdot t)}_{\text{area of wall}} = p \cdot \underbrace{\frac{\pi D^2}{4}}_{\text{area of pressure}}$$

Then $$\boxed{\sigma_l = \frac{pD}{4t}}$$ or $\frac{1}{2}\sigma_h = \sigma_l$

$\therefore \boxed{\sigma_h = 2\sigma_l}$



In problems, usually the pressure "p" is calculated when failure in the cylinder occurs. The yield strength "f_y" is given for the material of the cylinder.

Examples:

1. A cylinderical boiler of 2m in diameter & 4mm in wall thickness is under an interior pressure. The yield strength of the material is 310 MPa. Calculate the interior pressure to cause failure of the boiler. Use various theories of failure.
نظريات

Solution:-

First find the three principal stresses. The maximum principal stress is the hoop normal stress,

$$\sigma_1 = \sigma_h = \frac{PD}{2t} = \frac{P*2000}{2*4} = 250\,p$$

The intermediate principal stress is the longitudinal normal stress, $\sigma_2 = \sigma_l = \frac{1}{2}\sigma_h = 125p$

The minimum principal stress is the radial normal stress
$\sigma_3 = \sigma_r = 0$ (very small). لأن: هذا الاجهاد العمودي (الشعاعي) يكون في الجدران وبشكل قليل

i) By the Theory of maximum principal stress ($\sigma_1 = f_y$). Thus;

$$250p = 310 \quad , \text{This gives } p = \frac{310}{250} = 1.240\,MPa$$

* (Rankine) theory depends on max. stress only? in this pressure, the boiler its fail.

ii) By the Theory of max. Shearing stress ($\sigma_1 - \sigma_3 = f_y$),
Thus;

$$250p - 0 = 310 \quad \text{, this gives} \quad p = 1.240\ MPa$$

* in the case of $\sigma_3 \neq 0 \Rightarrow$ the pressure is different?

iii) By the Theory of max. energy of distortion (or shear)

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\, f_y^2$$

Thus;

$$(250p - 125p)^2 + (125p - 0)^2 + (0 - 250p)^2 = 2 * (310)^2$$

$$p^2 (125^2 + 125^2 + 250^2) = 192\,200$$
$$93750\, p^2 = 192\,200$$

Thus, $p = 1.431\ MPa$

* take the min. value of "p" for safety? H.W.

discuss the 3 difference value

الوعية الخزانات الكروية

2. Spherical vessels:

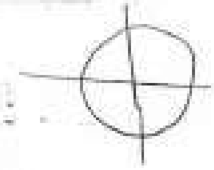
حمل القطر Consider a spherical vessel of mean diameter $\underline{D}$ & small thickness $\underline{t}$. Let $\underline{p}$ be the interior pressure.

Due to symmetry of all sections through diameters, the hoop normal stress is given by:

internal force   external force

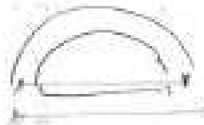
$$\sigma_h \cdot \underbrace{(\pi D \cdot t)}_{A} = p \cdot \left(\frac{\pi}{4} \cdot D^2\right)$$

Thus, $\boxed{\sigma_h = \frac{pD}{4t}}$

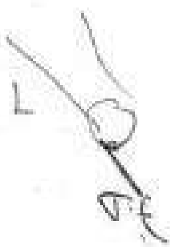


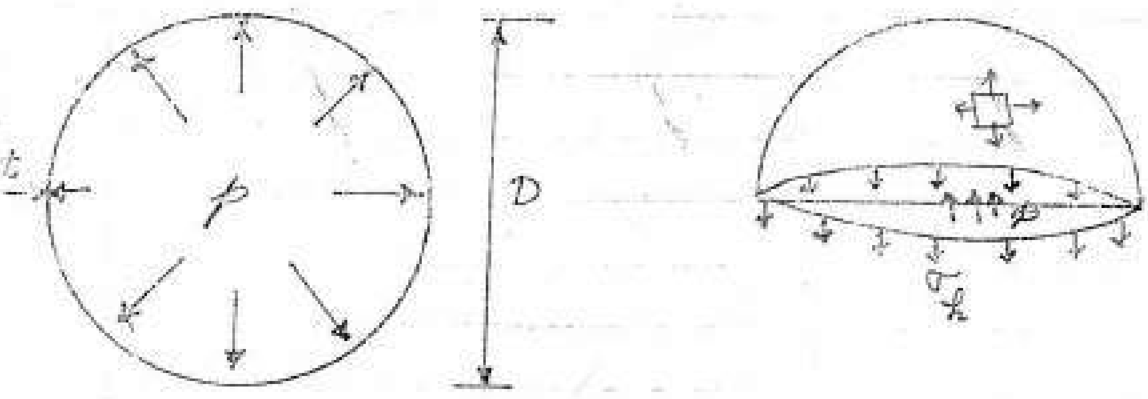
Thus here, $\sigma_1 = \sigma_h = \frac{pD}{4t}$

$$\sigma_2 = \sigma_h = \frac{pD}{4t}$$



$$\sigma_3 = \sigma_r = 0 \quad (\text{Small})$$





* at any section → $\frac{1}{2}$ spherical

1. A spherical vessel of mean diameter 3m & wall thick. 5mm will be filled by a gas. Calculate the max. safe pressure. Use various theories of failure. The yeild strength is 275 MPa.

Solution: Here;

$$\sigma_1 = \sigma_2 = \sigma_h = \frac{p * 3000}{4 * 5} = 150\,p$$

$$\sigma_3 = \sigma_r = 0$$

i) By $\sigma_1 = \sigma_{fy}$

This gives, $150p = 275 \Rightarrow p = 1.833\,MPa$.

ii) By $\cancel{\frac{1}{2}}(\sigma_1 - \sigma_3) = \cancel{\frac{1}{2}} f_y$

this gives $\frac{1}{2}(150p - 0) = \frac{1}{2} * 275$

Thus, $p = [1.833\,MPa]$.

iii) By $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2 f_y^2$

This gives;

$$(150p - 150p)^2 + (150p - 0)^2 + (0 - 150p)^2$$

$$= 2(275)^2$$

or $p = 1.833\,MPa$

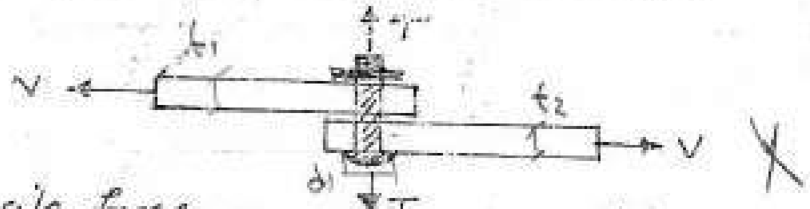
* Incase of $\sigma_1 = \sigma_2 = \sigma_3$ → the last equ. gave a trivial

Discuss the results

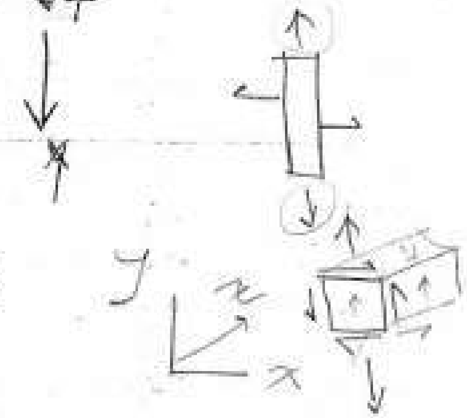
solution → "0" (hydrostatic pressure model) ?

this is one of disadvantages of the 3rd theory ?
واحده من مساوئ Von mises

3. A bolt or rivet under combined tensile & shearing forces.



Let "T" be the tensile force & the shearing force on a bolt or rivet of mean diameter D.



The tensile stress is; $\sigma_x = \dfrac{T}{\frac{\pi}{4} D^2}$

The shearing stress;

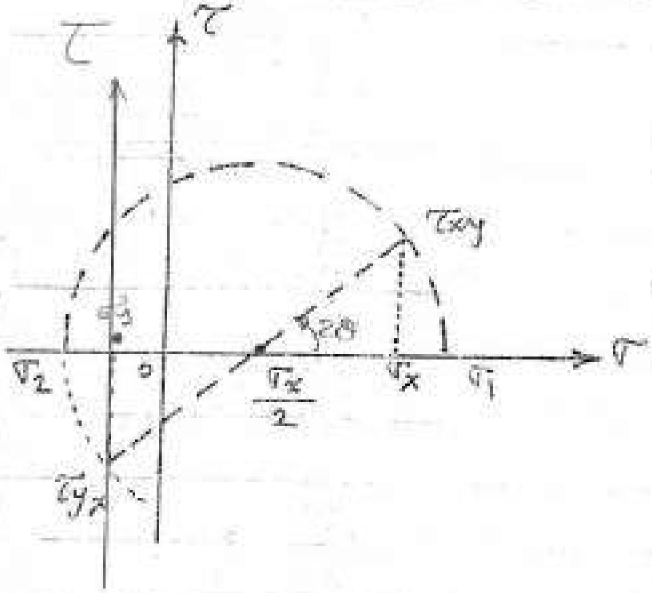
$$\tau_{xy} = \frac{V}{\frac{\pi}{4} D^2}$$

The other stresses are zero.

First find the three principal stresses (σ_1, σ_2 & σ_3). Here, the stresses are two dimensional (as $\sigma_y = \sigma_z = 0$ & $\tau_{yz} = \tau_{zx} = 0$).

By Mohr circle (or by formula):

$$\sigma_{1,2} = \frac{\sigma_x}{2} \mp \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} \quad , \text{Also } \sigma_3 = 0 .$$



Examples:

1. A bolt of mean dia. 16mm is under a tensile force 40kN. Find the safe shearing force on the bolt. Yield strength is 410 MPa. Use various theories of failure?

Solution: The tensile stress is, $\sigma_x = \dfrac{T}{\frac{\pi D^2}{4}}$

$$= \frac{4 \times 40 \times 10^3}{\pi (16)^2}$$

$$= 198.943 \text{ MPa}$$

The shearing stress;

$$\tau_{xy} = \frac{V}{\frac{\pi}{4} D^2} \quad \text{is unknown.}$$

Find the principal stresses; τ_y

$$\sigma_{1,2} = \frac{198.943}{2} \mp \sqrt{\left(\frac{198.943}{2}\right)^2 + \tau_{xy}^2}$$

Thus, $\sigma_1 = 99.472 + \sqrt{(99.472)^2 + \tau_{xy}^2}$

$\sigma_2 = 99.472 - \sqrt{(99.472)^2 + \tau_{xy}^2}$

$\sigma_3 = 0$

i) By $\sigma_1 = f_y$ Rankine

Thus, $99.472 + \sqrt{(99.472)^2 + \tau_{xy}^2} = 410$

This gives;

$\boxed{\tau_{xy} = 294.164 \text{ MPa}}$

The shearing force is,

$V = 294.164 * \frac{\pi}{4} * (16)^2$

$V = 59145 N$

$V \boxed{= 59.145 \text{ kN}}$

ii) By $\frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2} f_y$, or $\sigma_1 - \sigma_3 = f_y$

Thus, $\underbrace{\left\{ 99.472 + \sqrt{(99.472)^2 + \tau_{xy}^2} \right\}}_{\sigma_1} - \underbrace{\{0\}}_{\sigma_3} = 410$

This gives,

$\boxed{\tau_{xy} = 294.164 \text{ MPa}}$ & $\boxed{V = 59.145 \text{ kN}}$

* (The same Ans. because $\sigma_3 = 0$).

iii) By $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2 f_y^2$

substitute;

$$\left\{\left[99.472+\sqrt{(99.472)^2+\tau_{xy}^2}\right]-\left[99.472-\sqrt{(99.472)^2+\tau_{xy}^2}\right]\right\}^2$$

$$+\left\{\left[99.472-\sqrt{(99.472)^2+\tau_{xy}^2}\right]-[0]\right\}^2+\left\{[0]\right.$$

$$\left.-\left[99.472+\sqrt{(99.472)^2+\tau_{xy}^2}\right]\right\}^2=2*410^2$$

To solve, write $K=\sqrt{(99.472)^2+\tau_{xy}^2}$

Then; $(2K)^2+(99.472-K)^2+(99.472+K)^2=2*410^2$

Thus, $4K^2+(9895.076-198.948K+K^2)+(9895.076$
$+198.948K+K^2)=336200$

$6K^2=316409.850$, or $K=229.640$

resubstitute;

$$52734.975=(99.472)^2+\tau_{xy}^2$$

Then,

$$\boxed{\tau_{xy}=206.978\ MPa}$$

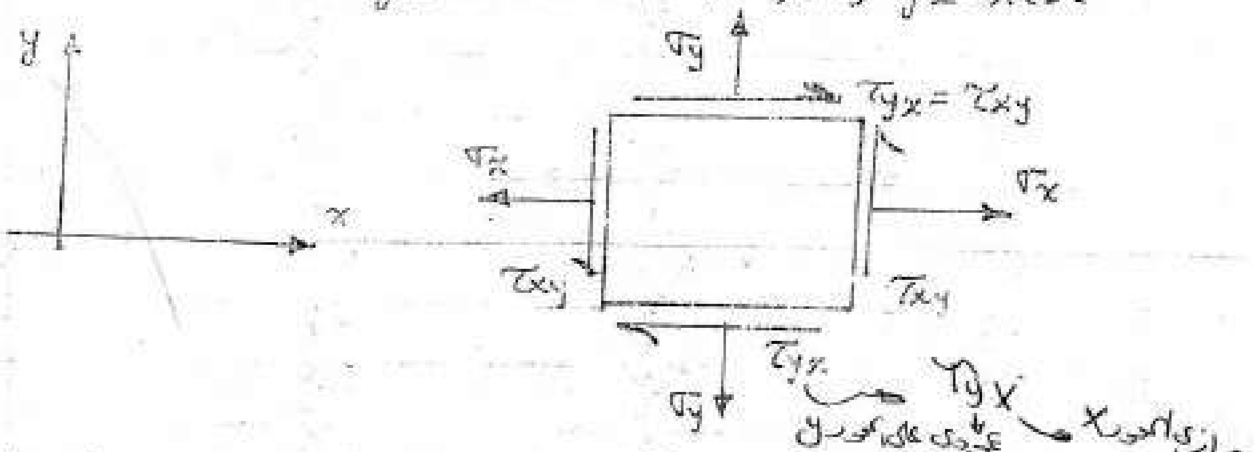
The shearing force V is;

$$V=206.978*\frac{\pi}{4}*(16)^2$$
$$\boxed{=41.615\ kN}$$

1.10* check the solution & study the difference bet. results ??

Two – Dimensional Problems of Elasticity In Cartesian Coordinates مسطحة

Consider a rectangular element in x & y – axes.



The stresses are the two normal stresses σ_x & σ_y & the shearing stresses $\tau_{xy} = \tau_{yx}$. The strains are the two normal strains ϵ_x & ϵ_y & the shearing strains $\gamma_{xy} = \gamma_{yx}$. $+\epsilon_z$

Use superposition & get:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 2(1+\nu)\frac{\tau_{xy}}{E}$$

$\eta_1 =$

$$\boxed{\epsilon_z = -\nu\,\epsilon_x - \nu\,\epsilon_y}$$

العلاقة العكسية

The inverse relation is:

لاحظ عادة

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$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\epsilon_x + \nu\,\epsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\epsilon_y + \nu\,\epsilon_x) \\ \tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \end{aligned}\right\} \text{"plane stresses"}$$

These are plane stresses. Notice that strain ϵ_z exists;

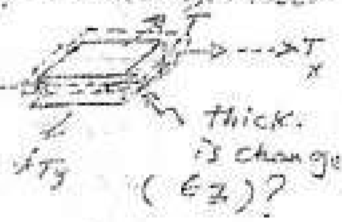
$$\epsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

; So the strains are three

$\epsilon_x, \epsilon_y, \epsilon_z$

– dimensional?

* when the body subjected to pull from 2-dirs? Thus the other 3rd dir is change → ϵ_z?



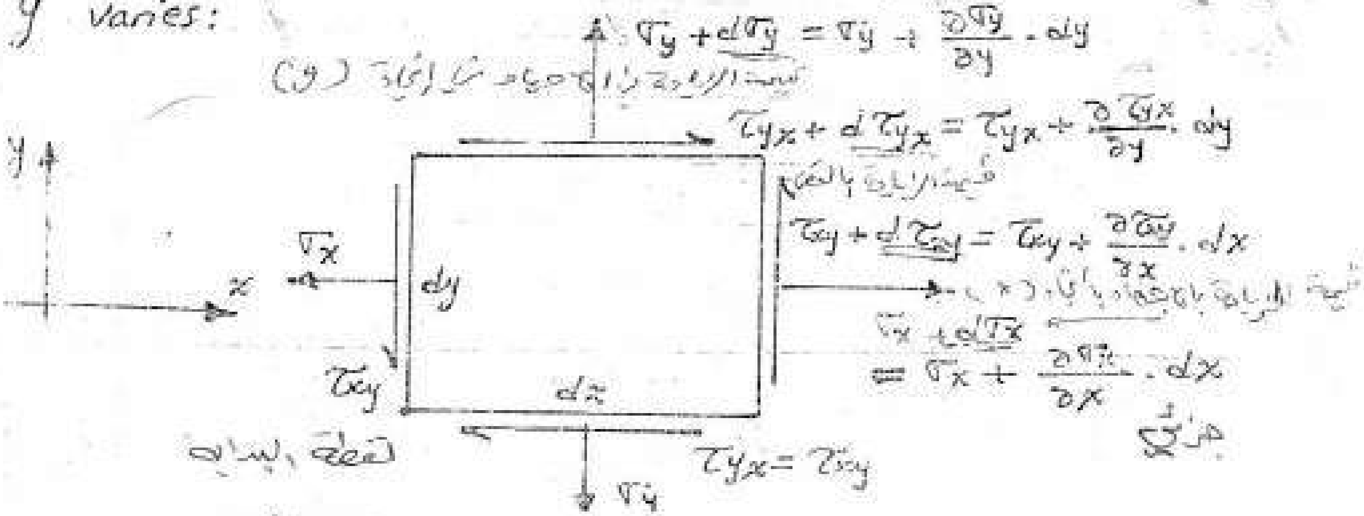
لو المسطح اتشد من اتجاهين فالسمك يتغير (ϵ_z) موجودة

الاجهادات توازن

Warping التعاج

Equilibrium of stresses:

when the stresses σ_x, σ_y & τ_{xy} vary in a plate, then these stresses should satisfy equilibrium. For this reason, consider the variation of stresses as x & y varies:



نقطة البداية

origen

τ_{xy}

X الاتجاه العمودي على

y والموازي

* لابد من وجود تغير في الإجهاد ولكن بشرط أن ينطبق عليه التوازن مثلما يتغير الانفعال بشرط التوافق.

$$\underbrace{\underset{①}{\sigma_y} + \frac{\partial \sigma_y}{\partial y} \cdot dy}_{②\ \text{increasing in } \sigma_y\text{-value}}$$

$\frac{\partial \sigma_y}{\partial y}$: slope in y-dir.

dy ← domain length (or range length)

Let "$\underline{t}$" be the thickness of the element $(dx \cdot dy)$.

use equilibrium in x – direction: ليكن (t) هو سمك العنصر في اتجاه (z) توازن قوى:

$\xrightarrow{+} \Sigma F_x = 0$

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \cdot dx\right)(dy \cdot t) - \sigma_x\,(dy \cdot t) + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot dy\right)(dx \cdot t)$$

$$- \tau_{yx} \cdot (dx \cdot t) = 0$$

simplify, then;

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \quad \text{------} \quad (1)$$

Also use equilibrium in y – direction;

$+\uparrow \Sigma F_y = 0$

$$\left(\sigma_y + \frac{\partial \sigma_y}{\partial y} \cdot dy\right)(dx \cdot t) - \sigma_y \cdot (dx \cdot t) + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \cdot dx\right)$$

$$\cdot (dy \cdot t) - \tau_{xy} \cdot (dy \cdot t) = 0$$

simplify, then;

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad \text{-----} \quad (2)$$

There are $\underline{\underline{3}}$ unknown stresses (σ_x, σ_y & $\tau_{xy} = \tau_{yx}$). But there are only $\underline{\underline{2}}$ equations of equilibrium. The problem is statically indeterminate. Find another equation.

* The compatibility of strains gives the additional equation.

$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

or
$$\frac{\partial^2}{\partial x^2}\left(\frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{\tau_{xy}}{G}\right) \quad \text{----- (3)}$$

These $\underline{\underline{3}}$ equs are difficult to solve.
* is it possible to solve 3-equs in math. ? try?

" STRESS FUNCTION " H.W.

An easier method of solution is to use a stress function $\phi(x, y)$. Define the stresses as:

$$\left\{ \begin{aligned} \sigma_x &= \frac{\partial^2 \phi}{\partial y^2} \\ \sigma_y &= \frac{\partial^2 \phi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \right\}$$

These definition satisfy completely the equs of equilibrium. Substitute into the third (or compatibility) equation, then;

$$\frac{\partial^2}{\partial x^2}\overbrace{\left\{ \overset{\sigma_y}{\frac{\frac{\partial^2\phi}{\partial x^2}}{E}} - \nu \overset{\sigma_x}{\frac{\frac{\partial^2\phi}{\partial y^2}}{E}} \right\}}^{\epsilon_y} + \frac{\partial^2}{\partial y^2}\overbrace{\left\{ \overset{\sigma_x}{\frac{\frac{\partial^2\phi}{\partial y^2}}{E}} - \nu \overset{\sigma_y}{\frac{\frac{\partial^2\phi}{\partial x^2}}{E}} \right\}}^{\epsilon_x}$$

$$= \frac{\partial^2}{\partial x \partial y}\left\{ \frac{\overset{\tau_{xy}}{-\frac{\partial^2\phi}{\partial x \partial y}}}{\frac{E}{2(1+\nu)}} \right\}$$

simplify, then:

$$\boxed{\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0}$$

Define the operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Then the basic (or governing) equation is:

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0$$

or: $\nabla^2 \cdot \nabla^2 \phi = 0$ or $\boxed{\nabla^4 \phi = 0}$

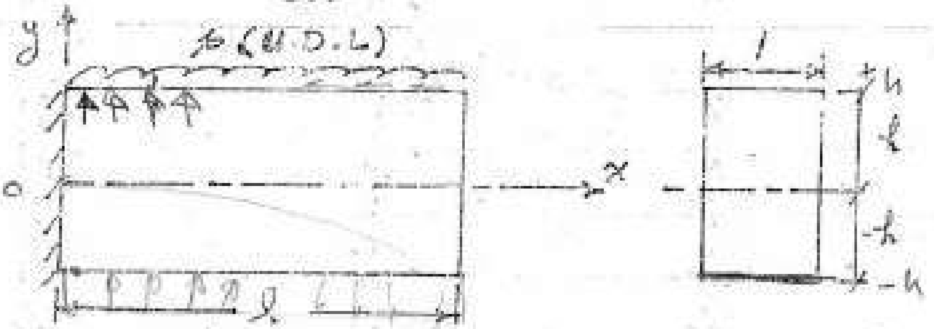
This equation is difficult to solve. The method of trials is usually used. The exact solution must satisfy $\nabla^4\phi = 0$ & all boundary conditions.

<u>Examples</u> (2 - D problemes in x & y):

1. For the below loaded cantilever, check the following stress function $\underline{\phi(x,y)} = \frac{P}{40h^3}(-10h^3x^2 + 30h^2lxy$

$$- 15h^2x^2y + 5l^2y^3 + 2l^2y^3 - 10lxy^3 + 5x^2y^3 - y^5)$$

Then study the stresses & deflections in this Cantilever (using $u = v = \frac{dv}{dx} = 0$ at $x = y = 0$).



solution: <u>How to start?</u>

First check that the given stress function satisfies $\nabla^4\phi = 0$. (or find the stresses σ_x, σ_y & $\tau_{xy} = \tau_{yx}$ & check the equilibrium).

Here use, $\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$

Thus: $\frac{\partial^4\phi}{\partial x^4} = \frac{P}{40h^3}(0 + 0 - - - - + 0) = 0$

$$\frac{\partial^4\phi}{\partial x^2\partial y^2} = \frac{P}{40h^3}(0 + 0 + --- -120y) = \frac{-120Py}{40h^3}$$

Point ; $\dfrac{\partial^4 \phi}{\partial x^2 \partial y^2} = \dfrac{p}{40 h^3} (0 + 0 + \text{----} + 60y \text{--} 0) = \dfrac{60py}{40h^3}$

Add: $0 + 2 \times \dfrac{60py}{40h^3} + \dfrac{-120py}{40h^3} = 0$ (o.k.)

it satisfy checks

Next find the stresses & check the boundary conditions:

$\underline{\sigma_x} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{p}{40h^3} (30 l^2 \underline{\underline{y}} + 12 h^2 \underline{\underline{y}} - 60 l x \underline{\underline{y}} + 30 x^2 \underline{\underline{y}} - 20 \underline{\underline{y^3}})$

$\sigma \not\propto y$

* Notice that direct proportion of σ w.r.t. y? the section is not remain plane? Compare with strength of material assumptions?

(Notice here that σ_x is not linear in "y" as given in strength of materials ($\sigma_x = \dfrac{M}{I} \cdot y$)?

warping

$\underline{\sigma_y} = \dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{p}{40h^3} (-20h^3 - 30h^2 y + 10y^3)$

(Notice here that σ_y cannot be found from strength of materials).

$\underline{\tau_{xy}} = \tau_{yx} = -\dfrac{\partial^2 \phi}{\partial x \partial y} = -\dfrac{p}{40h^3} (30h^2 l - 30h^2 x - 30 l y^2 + 30 x y^2)$

Now come to the boundary conditions.

i) At top face ($\underline{y = +h}$ for all x):

$\sigma_y = -p$ (applied).

$\tau_{xy} = 0$ (no shearing stress).

check;

$$(\sigma_y)_{y=+h} = \frac{p}{40h^3}(-20h^3 - 30h^2 \cdot h + 10h^3)$$

$$= -p \quad (\text{o.k.})$$

$$(\tau_{xy})_{y=+h} = \frac{-p}{40h^3}(\cancel{30h^2 l} - \cancel{30h^2 x} - \cancel{30 l h^2} + \cancel{30 x h^2})$$

$$= 0 \quad (\text{o.k.})$$

ii) At bottom face ($y = -h$ for all x):
$\sigma_y = 0$ & $\tau_{xy} = 0$ (free face)

Here, $$(\sigma_y)_{y=-h} = \frac{p}{40h^3}(-20h^3 + 30h^2 \cdot h - 10h^3)$$

$$= 0 \quad (\text{o.k.})$$

$$(\tau_{xy})_{y=-h} = -\frac{p}{40h^3}(30h^2 l \mp 30h^2 x \mp 30 l h^2 \mp 30 x h^2)$$

$$= 0 \quad (\text{o.k.})$$

iii) End face ($x = l$ for all y)

Here, $$\left.\begin{array}{l}\sigma_x = 0 \\ \tau_{xy} = 0\end{array}\right\} \text{(free face)}$$

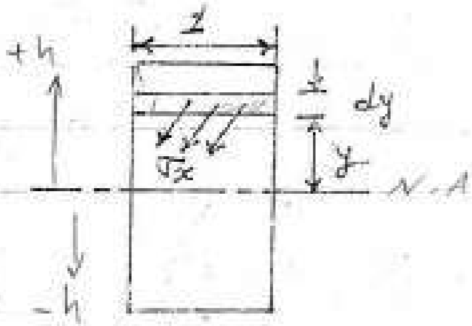
Thus;

$$(\sigma_x)_{x=l} = \frac{p}{40h^3}(30l^2y \mp 12h^2y \pm 60ly^2 \mp 30l^2y \pm 20y^3)$$

$$= \frac{P}{40h^3}(12h^2y - 20y^3) \neq 0 \quad \underline{\text{(Not o.k.)}}$$

Use <u>Saint-venant principal</u>, where the resultant force & the resultant moment must satisfy the boundary conditions.

The normal axial force is:



$$N = \int_{-h}^{+h} \sigma_x \cdot \overbrace{(1 \cdot dy)}^{arm}$$

force

$$N = \int_{-h}^{+h} \frac{P}{40h^3}(12h^2y - 20y^3)dy$$

$$= \frac{P}{40h^3}\left| \frac{12h^2y^2}{2} - \frac{20y^4}{4} \right|_{-h}^{+h}$$

$$= 0 \quad (\text{o.k.})$$

The resultant of moment is:

$$M = \int_{-h}^{+h} y \cdot \sigma_x \cdot (1 \cdot dy)$$

$$M = \int_{-h}^{+h} y \cdot \frac{P}{40h^3}(12h^2y - 20y^3) \cdot dy$$

$$M = \frac{P}{40h^3}\int_{-h}^{+h}(12h^2y^2 - 20y^4)\,dy$$

$$= \frac{P}{40h^3} \left| \frac{12h^2y^3}{3} - \frac{20y^5}{5} \right|_{-h}^{+h} = 0 \qquad (o.K.)$$

Thus all stresses are exact.

$$\sigma_x = \frac{P}{40h^3}(30l^2y + 12h^2y - 60lxy + 30x^2y - 20y^3)$$

$$\sigma_y = \frac{P}{40h^3}(-20h^3 - 30h^2y + 10y^3)$$

$$\tau_{xy} = \tau_{yx} = -\frac{P}{40h^3}(30h^2l - 30h^2x - 30ly^2 + 30xy^2)$$

Then find the strains:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{P}{40h^3E}\left[(30l^2y + 12h^2y - 60lxy + 30x^2y - 20y^3) - \nu(-20h^3 - 30h^2y + 10y^3)\right]$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} = \frac{P}{40h^3E}\left[(-20h^3 - 30h^2y + 10y^3) - \nu(30l^2y + 12h^2y - 60lxy + 30x^2y - 20y^3)\right]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$\gamma_{xy} = -\frac{2(1+\nu)P}{40h^3E}(30h^2l - 30h^2x - 30ly^2 + 30xy^2)$$

For simple calculations, let $\nu = 0$, then;

$$\epsilon_x = \frac{P}{40h^3E}(30l^2y + 12h^2y - 60lxy + 30x^2y - 20y^3)$$

$$\epsilon_y = \frac{P}{40h^3E}(-20h^3 - 30h^2y + 10y^3)$$

$$\gamma_{xy} = -\frac{P}{20h^3E}(30h^2l - 30h^2x - 30ly^2 + 30xy^2)$$

Start from $\epsilon_x = \frac{\partial u}{\partial x}$ & integrate. Then;
I.w.r.to x

$$u = \frac{P}{40h^3E}(30l^2yx + 12h^2yx - 30lx^2y + 10x^3y - 20y^3x) + f(y)$$

Also use $\epsilon_y = \frac{\partial v}{\partial y}$ & integrate. Then;
I.w.r.to y

$$v = \frac{P}{40h^3E}\left(-20h^3y - 15h^2y^2 + \frac{10}{4}y^4\right) + g(x)$$

use, $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$ → to find Constants?

$$-\frac{P}{20h^3E}(30h^2l - 30h^2x - 30ly^2 + 30xy^2)$$

$$= \left\{\frac{P}{40h^3E}(0+0+\cdots+0) + \frac{dg}{dx}\right\} + \left\{\frac{P}{40h^3E}(30l^2x + 12h^2x - 30lx^2 + 10x^3 - 60y^2x) + \frac{df}{dy}\right\}$$

Rearrange: $+\frac{P}{20h^3E}(30h^2l - 30h^2x) + \frac{dg}{dx} + \frac{P}{40h^3E}(30l^2x + 12h^2x - 30lx^2 + 10x^3) = -\frac{P}{20h^3E}\times(-30ly^2) - \frac{df}{dy}$

write : $\boxed{\frac{P}{20h^3E}(30\,ly^2) - \frac{df}{dy} = C}$

$$\therefore \frac{P}{20h^3E}(30h^2l - 30h^2x) + \frac{dg}{dx} + \frac{P}{40h^3E}(30l^2x$$

$$+ 12h^2x - 30lx^2 + 10x^3) = C$$

Thus find: $f = \frac{P}{20h^3E}\left(30\frac{ly^3}{3}\right) - Cy + K_1$

And;

$$g = -\frac{P}{20h^3E}\left(30h^2xl - 30h^2\frac{x^2}{2}\right) - \frac{P}{20h^3E}\left(30l^2\frac{x^2}{2}\right.$$

$$\left. + 12h^2\frac{x^2}{2} - 30l\frac{x^3}{3} + 10\frac{x^4}{4}\right) + Cx + K_2$$

substitute, then $u = ---$

H.W. $v = ---$

$\frac{\partial v}{\partial x} = ---$, $\frac{\partial u}{\partial y} = ---$

Use $u = v = \frac{\partial v}{\partial x} = 0$ at $x = y = 0$ & find $K_1 = K_2 = 0$

& $C = \frac{3}{2}\left(\frac{PL}{Eh}\right)$

The deflection of the cantilever is given;

$(v)_{\substack{y=0\\x=0}} =$ zero , At the free end, the deflection is:

$$(v)_{\substack{y=0\\x=l}} = \text{----}$$

(Ans.)

From strength of materials, this end deflection is:

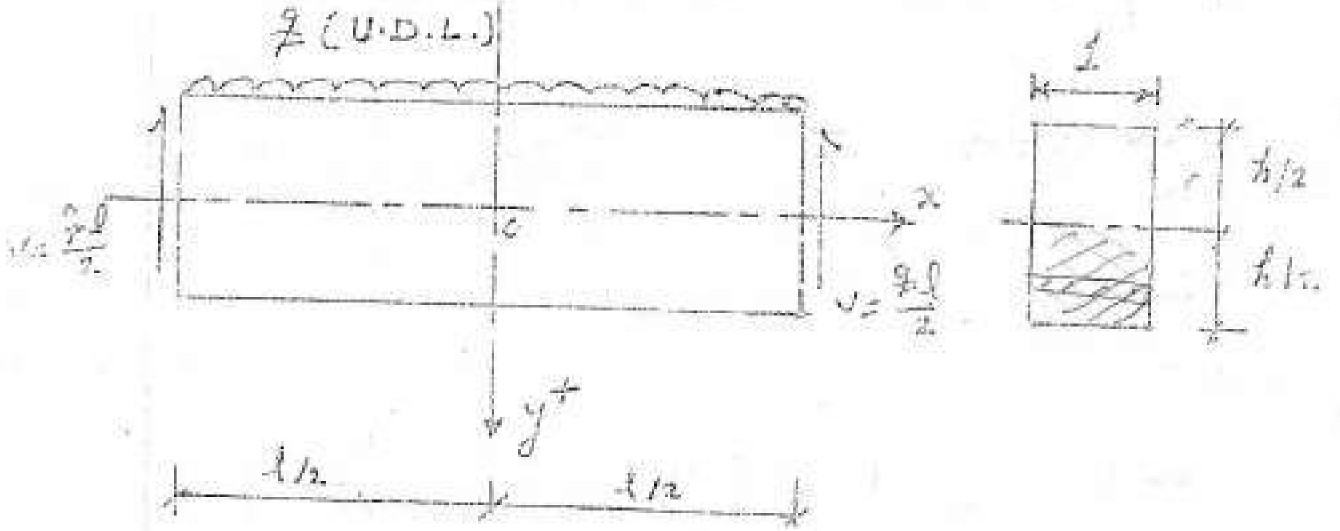
$$\Delta = \frac{Pl^4}{8EI} \text{; where } I = \frac{1}{12}\cdot 1\,(2h)^3 = \frac{2}{3}h^3 .$$

عبدالرحمن عبدالله أحمد

Dec., 28, 2011

Dr. Mayeer ... 74

2. A beam is simply supported by end shearing forces.



The stresses are: $\sigma_x = \frac{12q}{h^3}\left[\frac{1}{2}\left(\frac{l^2}{4} - x^2\right)y - \frac{h^2}{20}y + \frac{1}{3}y^3\right]$

$$\sigma_y = -\frac{6q}{h^3}\left(\frac{1}{3}y^3 - \frac{h^2}{4}y + \frac{h^3}{12}\right)$$

$$\tau_{xy} = \tau_{yx} = -\frac{6q}{h^3}\left(\frac{h^2}{4} - y^2\right)x$$

check the stresses?

Hint: First check the equilibrium of stresses:

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0 \end{aligned}\right\}$$ equil. equs. (2-equs)?

Next check the stresses on boundary surfaces:

i) $\sigma_y = \tau_{xy} = 0$ at $y = +\frac{h}{2}$ for all x (bottom face)

(free surfaces)

$\left\{\begin{array}{l} -ve \rightarrow \text{comp.} \\ +ve \rightarrow \text{tens.} \end{array}\right\}$ for surfaces

ii) $\left.\begin{array}{l} \sigma_y = -q \\ \tau_{xy} = 0 \end{array}\right\}$ at $y = -\frac{h}{2}$ for all x (top face)

iii) $\sigma_x = 0$ at $x = \frac{l}{2}$ for all y (end face)

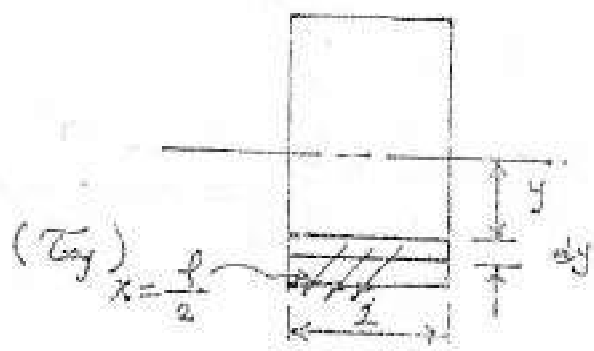
If not o.k., use:

$$N = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \, (1 \cdot dy) = 0$$

$$M = \int_{-\frac{h}{2}}^{+\frac{h}{2}} y \cdot \sigma_x \cdot (1 \cdot dy) = 0$$

Also check;

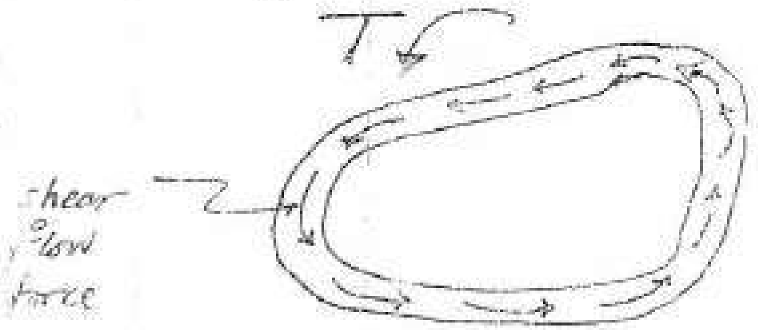
$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} (\tau_{xy})_{x=\frac{l}{2}} \cdot (1 \cdot dy) = \frac{-q\,l}{2}$$ (end shearing force)



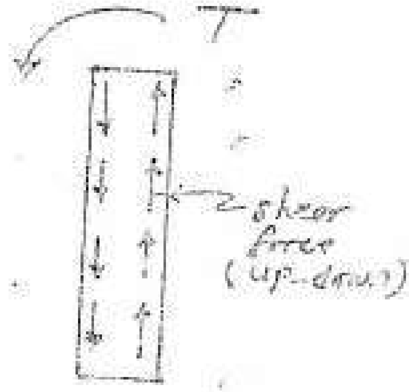
الإلتواء

TORSION

Torsion of closed thin-walled sections (or cells) & torsion of thin rectangular sections are important in civil engineering.



closed thin-walled section



thin rectangular section

closed thin-walled sections (or cells):

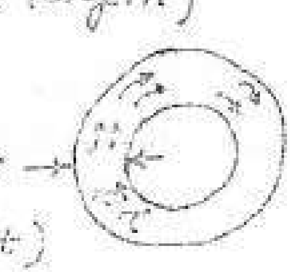
For this, a quantity (shear flow) is needed. Shear flow in any position in the thin wall of a closed cell is defined as:

shear flow = shear stress * thickness of wall

or

$$\boxed{q = \tau t}$$

(force / unit length)



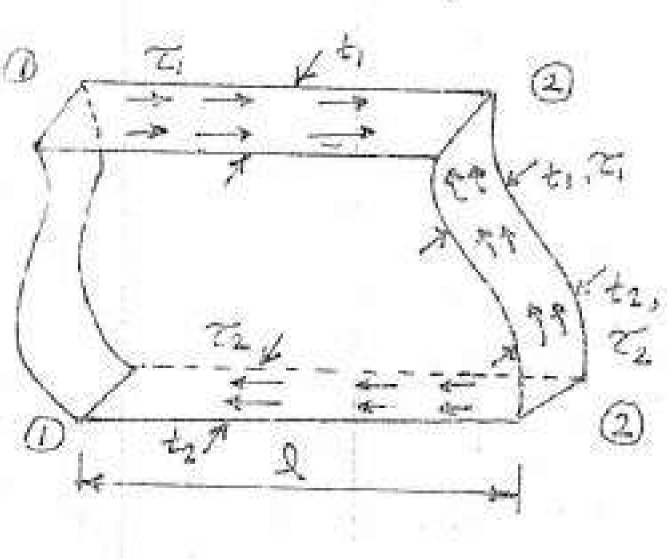
$(q = \tau \cdot t)$

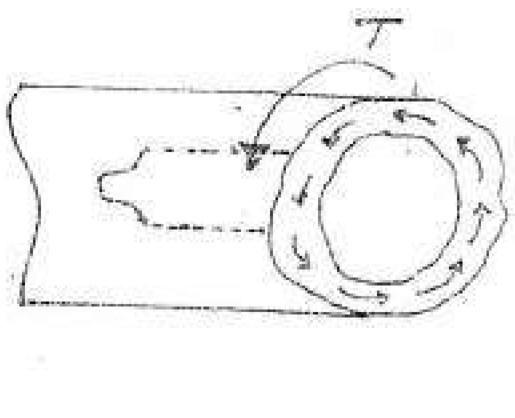
The following theorems are needed:

1. The shear flow is constant in any closed thin-walled section. The shearing stress "τ" & the thickness "t" may vary in the wall, but $q = \tau t$ must remain constant. This is analogous to the flow of water in closed channel (flow = velocity * section area).

Proof:

Take a portion cut out by longitudinal section & cross sections in a tube.





At edge 1, the shear flow is $q_1 = \tau_1 t_1$, At edge 2, the shear flow is $q_2 = \tau_2 t_2$. To prove $q_1 = q_2$, consider the horizontal (or longitudinal) equilibrium.

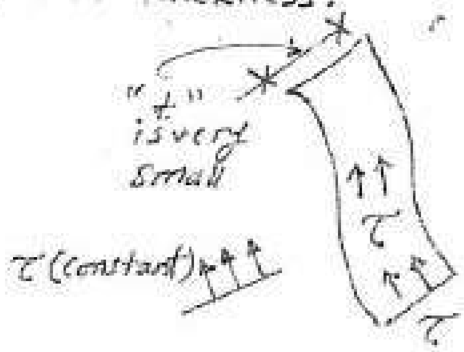
$$\tau_1 * t_1 * l = \tau_2 * t_2 * l$$

Thus the $\tau_1 * t_1 = \tau_2 * t_2$ or $q_1 = q_2$.

Important notes:

1) The shearing stress "τ" must be parallel to the edges of the wall.
(The component normal to the edge doesnot exist).

2) The shearing stress "τ" is constant across the small thickness (at any position). The thickness "t" is very small & τ cannot vary in this thickness.

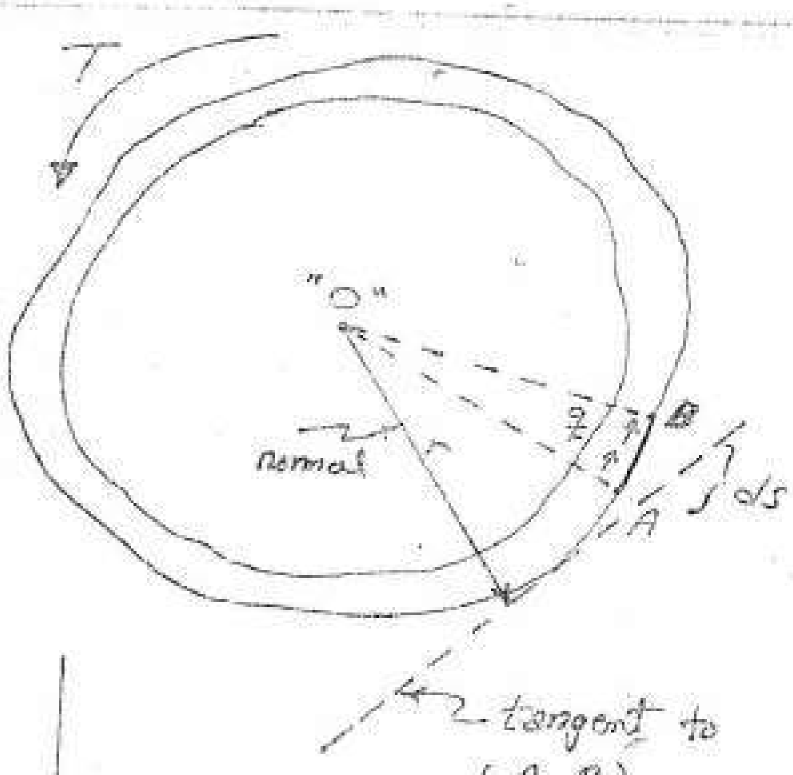


2. The torque "T" in the closed thin-walled section is;

$$\boxed{T = 2qA}$$

where "A" is the area enclosed by the wall of the section.

Take a small length $A-B = ds$ in the wall. The shear flow $q = \tau t$ in $A-B$ will produce torque.



$$dT = r.\tau.t.ds$$
$$= r.q.ds$$

where "r" is the normal to A-B from origin "O". The origin "O" can by point inside (or outside) the section.

The area of the triangle OAB is $dA = \frac{1}{2} r\,ds$, $ds = \frac{2dA}{r}$

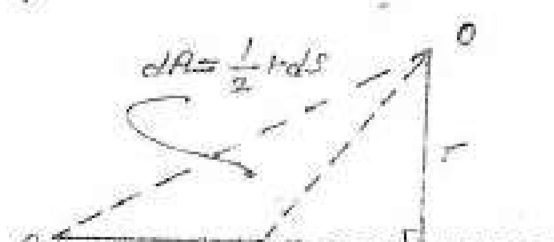
thus; $dT = 2q\,dA$.

$$T = \int 2q\,dA$$

$$= 2q \int dA \quad (\text{as } q \text{ is constant})$$

Thus;

$$\boxed{T = 2qA}$$



A = inclosed Area.
q = shear flow

3. The torsion Constant of a closed-thin-walled section is:

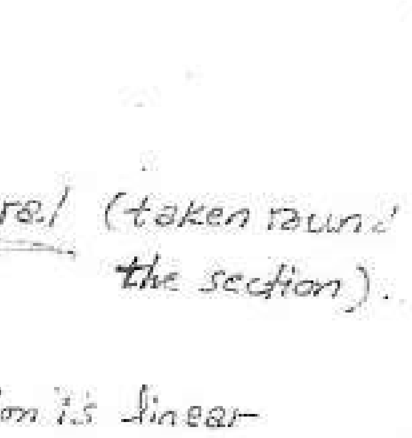
$$J = \frac{4A^2}{\int_c \frac{ds}{t}} \quad (\text{length}^4)$$

where $\int_c \frac{ds}{t}$ is the Contour integral (taken round the section).

Definition: The torque-twist relation is linear (Hook's law).

The constant $k = \frac{T}{\theta}$ is torsional stiffness.

This is the torque to produce unit angle of twist. Usually:



$$T = \frac{GJ}{l}\cdot\theta \quad \text{or}$$

$$k = \frac{GJ}{l} \quad (\text{when } \theta = 1) \text{ radian}$$

where "l" is the length of the tube & G the modulus of rigidity (or shear modulus of elasticity), & J is the torsion Constant of the section. The product GJ is called (torsional rigidity) of the section. Notice that $GJ \rightarrow EI$ in the bending of beams. Here, prove that

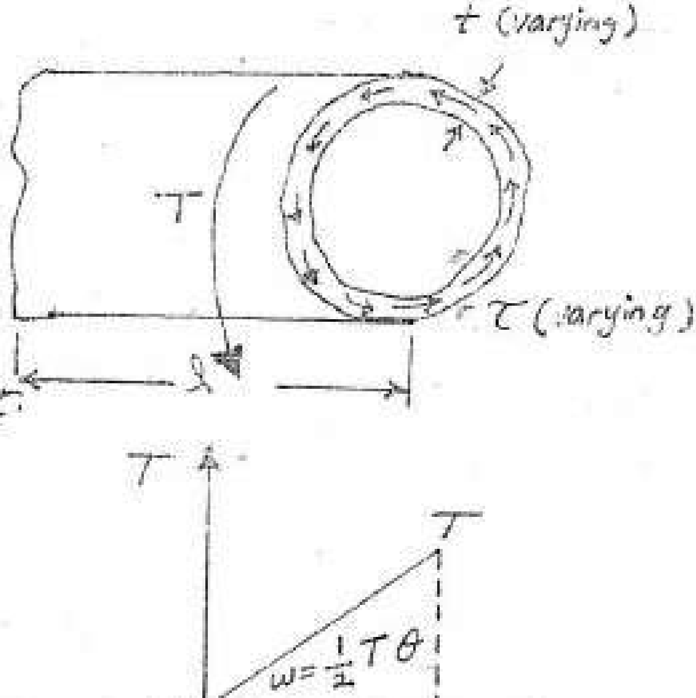
$$\boxed{J = \frac{4A^2}{\oint_c \frac{ds}{t}}}$$

<u>Proof</u>: Use the method of real work.

External work = Internal strain energy.

- (τ & $t \to$ varying)
- ($\tau * t$ = constant)

The external work is;



$$W = \frac{1}{2} T \theta$$

as ($T - \theta$) relation is <u>linear</u>.

The internal strain energy is:

$$U = \int_{Vol} \frac{\tau^2}{2G} \, d(vol) \qquad \text{(volume integral)}$$

as <u>τ</u> is the <u>only stress</u> in the tube.

$$\delta \frac{T}{G} \cdot \tau$$

Here: $d(vol) = l.t.ds$ (volume of a strip)

$$\tau = \frac{q}{t} = \frac{T/2A}{t}$$

substitute, $U = \int_c \frac{\left(\frac{T}{2At}\right)^2}{2G} \cdot l.t.ds$

$$= \int_c \frac{T^2 l}{8A^2 G t} ds = \frac{T^2 l}{8A^2 G} \int_c \frac{ds}{t}$$

Use $W = U$, then;

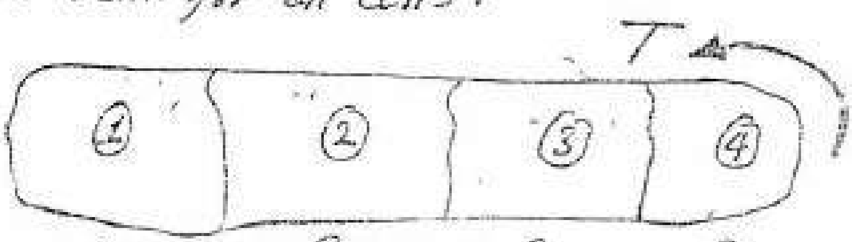
$$\frac{1}{2} T\theta = \frac{T^2 l}{8A^2 G} \int_c \frac{ds}{t}$$

or

$$T = \frac{4A^2}{\int_c \frac{ds}{t}} \cdot \frac{G}{l} \cdot \theta$$

Compare to $T = \frac{GJ}{l} \cdot \theta$, Then $\boxed{J = \frac{4A^2}{\int_c \frac{ds}{t}}}$

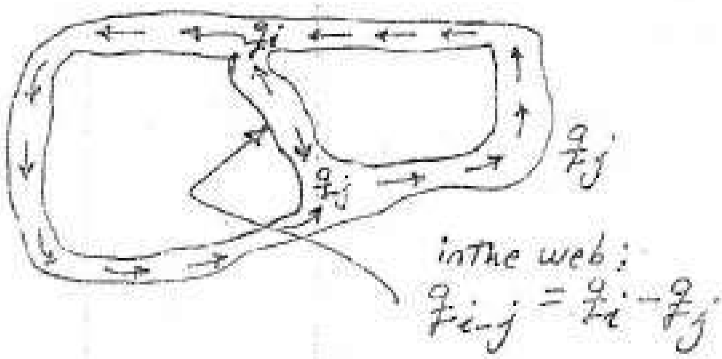
4. For multi-cell sections (statically indeterminate), the twist is the same for all cells.



$$\theta_1 = \theta_2 = \theta_3 = \theta_4$$

twisting angle equal in all cell

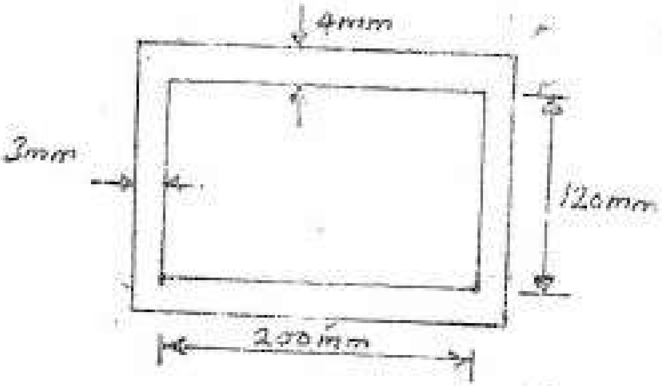
The shear flow in an interior web is :



$$q_{i-j} = q_i - q_j$$

Examples : (closed thin-walled cells)

1. A tube of length 800mm & rectangular cell (as shown) is under a torque 2 KN·m. Calculate the angle of twist & also the shearing stresses in the wall of the tube? $G = 80 \times 10^3$ MPa.
   90×10^3



solution.

First find the enclosed area,

$A = 200 \times 120 = 24000 \ mm^2$

use , $T = 2qA$

Then the constant shear flow is; $q = \dfrac{T}{2A}$

$$q = \frac{2 * 10^6 \, (N \cdot mm)}{2 * 24000 \, (mm^2)} = \frac{1000}{24} \; N/mm = 41.667 \; N/mm$$

The shearing stresses in the wall is; $\tau = \frac{q}{t}$

In the wall of 4mm: $\tau = \frac{41.667}{4} = 10.417 \; N/mm^2$

" " " " 3mm:

$$\tau = \frac{41.667}{3} = 13.889 \; N/mm^2$$

To find the angle of twist calculate J. Here;

$$J = \frac{4A^2}{\oint_c \frac{ds}{t}} \Rightarrow J = \frac{4 * (24000)^2}{2\left(\frac{200}{4} + \frac{120}{3}\right)} = 12.8 * 10^6 \; mm^4$$

use; $\theta = \frac{Tl}{GJ} = \frac{2 * 10^6 * 800}{80 * 10^6 * 12.8 * 10^6} = 1.563 * 10^{-3}$ radian

$= 1.553 * 10^{-3} * \frac{180}{\pi}$

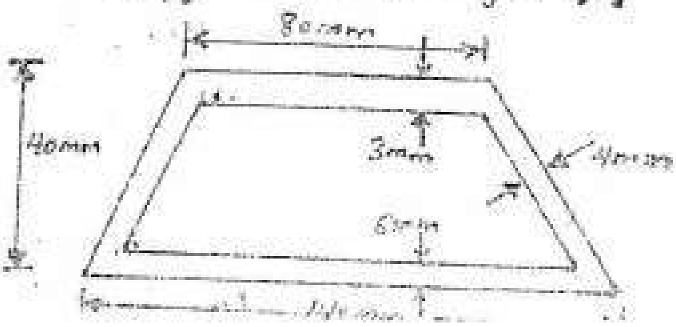
$= 0.08955^\circ$

H.W.

2. A tube of length 1.7m & the cross-section as shown. The tube is under a torque 2 kN.m & axial tensile force 25 kN. check the failure by Rankine & tresca theories. $G = 80 * 10^3 \; N/mm^2$ (not needed), $f_y = 410$ MPa (yield)?

Hint: Here $\sigma_x = \frac{P}{\text{area of wall}}$

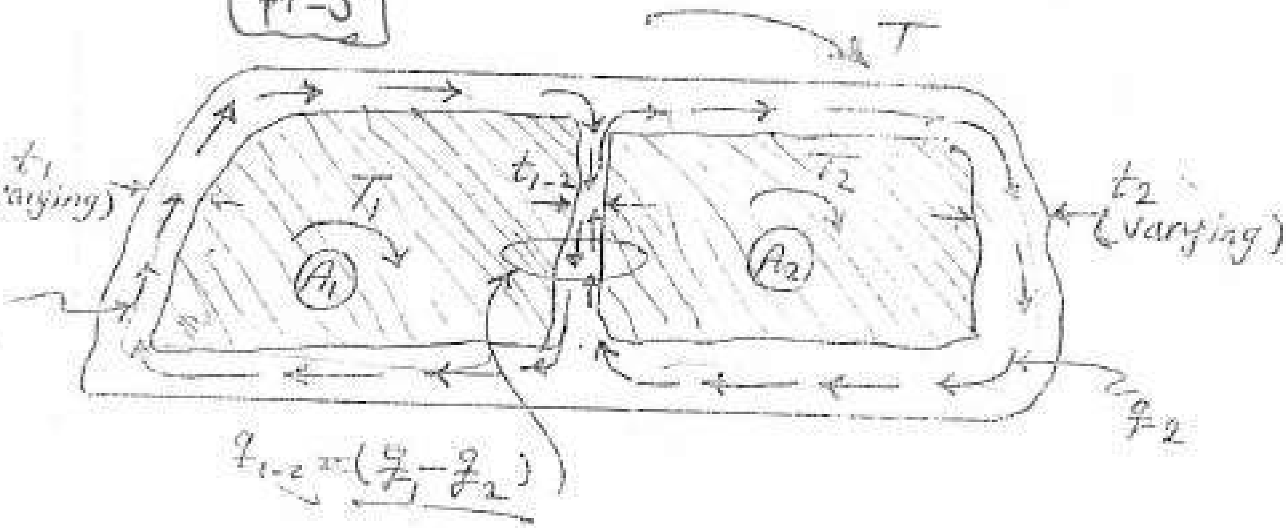
, $\sigma_y = \sigma_z = 0$



Torsion of thin-walled multi-cell sections (statically indeterminate)

Consider a tube of two cells under a torque T. Let q_1 be the shear flow of cell 1 (with enclosing area A_1 & thickness t_1) & let q_2 be the shear flow in cell 2 (with thickness t_2) & enclosing area A_2). The diaphragm has a shear flow $(q_1 - q_2)$ & thickness t_{1-2} دياجرام

$\boxed{q_{i-j}}$



The torque T_1 in cell 1 by the shear flow q_1 is:

$$T_1 = 2A_1 q_1$$

The torque T_2 in cell 2 by the shear flow q_2 is:

$$T_2 = 2A_2 q_2$$

Distorsion

* عن التربار الذي تكون عرضنا التراكيب اهتياجًا .

Equilibrium gives; $\quad T = T_1 + T_2$

or

$$T = 2A_1 q_1 + 2A_2 q_2 \quad \text{----} (1)$$

This is one equation with two unknowns (q_1 & q_2). The problem is statically indeterminate.

Next consider the twist θ_i in cell "i" where "i" is 1 or 2. To find θ_i, use the method of real work ($w = u$):

شغل خارجي

$$\frac{1}{2} T_i \theta_i = \left(\int_{vol} \frac{\tau^2}{2G} \, d(vol) \right)_{cell\ i}$$

= شغل داخلي

$$\frac{1}{2} T_i \underline{\underline{\theta_i}} = \left(\int_c \frac{\tau^2}{2G} \cdot \underbrace{l\, t\, ds}_{d(vol)} \right)_{cell\ \underline{\underline{i}}}$$

Use;

$$\frac{1}{2} (2A_i q_i) \theta_i = \left(\int_c \frac{q_i l}{2G} \cdot \tau \, ds \right)$$

(as $T_i = 2A_i q_i$ & $q_i = \tau_i t_i$)

$$\theta_i = \frac{l}{2GA_i} \left(\int_c \tau \, ds \right)_{cell\ i}$$

or $\boxed{\theta_i = \frac{l}{2GA_i} \left(\int_c q \frac{ds}{t} \right)_{cell\ i}}$ or $\frac{G\theta_i}{l} = \frac{1}{2A_i} \left(\int_c q \cdot \frac{ds}{t} \right)_{ce}$

Notice that castigliano's theorem can be used to find $\theta_i = \frac{\partial U}{\partial T_i}$.

Here, $$U = \left(\int_{vol} \frac{\tau^2}{2G} d(vol) \right)_{cell\, i} = \left(\int_c \frac{(\frac{q}{t})^2}{2G} \cdot l\, t\, ds \right)_{cell\, i}$$

$$= \left(\int_c \frac{q^2 l}{2G} \cdot \frac{ds}{t} \right)_{cell\, i} = \left(\int_c \frac{(\frac{T}{2A})^2 l}{2G} \cdot \frac{ds}{t} \right)_{cell\, i}$$

$$= \left(\int_c \frac{T^2 l}{8A^2 G} \cdot \frac{ds}{t} \right)_{cell\, i}$$

Then $$\theta_i = \frac{\partial U}{\partial T_i} = \left(\int_c \frac{2T l}{8A^2 G} \cdot \frac{ds}{t} \right)_{cell\, i}$$

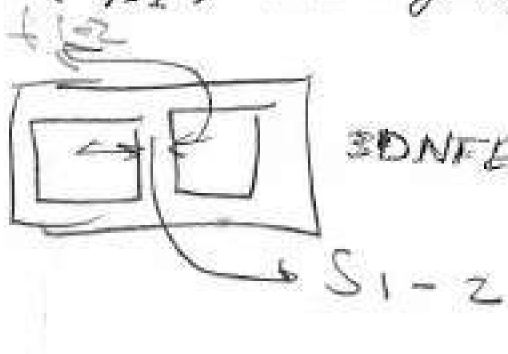
$$= \left(\int_c \frac{(2Aq) l}{4A^2 G} \cdot \frac{ds}{t} \right)_{cell\, i}$$

or

$$\boxed{\theta_i = \frac{l}{2A_i G} \left(\int_c q \cdot \frac{ds}{t} \right)_{cell\, i}}$$ (same resultant).

$\overline{\text{leap}}$ for cell 1 : $$\boxed{\frac{G\theta_i}{l} = \frac{1}{2A_1} \left[q_1 \cdot \Sigma \frac{S_1}{t_1} + (q_1 - q_2) \cdot \frac{S_{1-2}}{t_{1-2}} \right]}$$

(where (S_{1-2}) is the length of the diaphragm).



3DNFEA - AL-sharbaf - 1989 - England.

For cell 2 :

$$\boxed{\frac{G\theta_2}{l} = \frac{1}{2A_2}\left[q_2 \cdot \Sigma \frac{S_2}{t_2} + (q_2 - q_1) \cdot \frac{S_{1-2}}{t_{1-2}}\right]}$$

The compatibility condition gives:

$$\theta_1 = \theta_2 \quad \text{or} \quad \frac{G\theta_1}{l} = \frac{G\theta_2}{l}$$

or

$$\frac{1}{2A_1}\left[q_1 \cdot \Sigma \frac{S_1}{t_1} + (q_1 - q_2) \cdot \frac{S_{1-2}}{t_{1-2}}\right]$$

$$= \frac{1}{2A_2}\left[q_2 \cdot \Sigma \frac{S_2}{t_2} + (q_2 - q_1) \cdot \frac{S_{1-2}}{t_{1-2}}\right] \quad \text{-----}(2)$$

These two equations will gives q_1 & q_2. The method can be extended into tubes of three or more cells.

$$T = T_1 + T_2$$

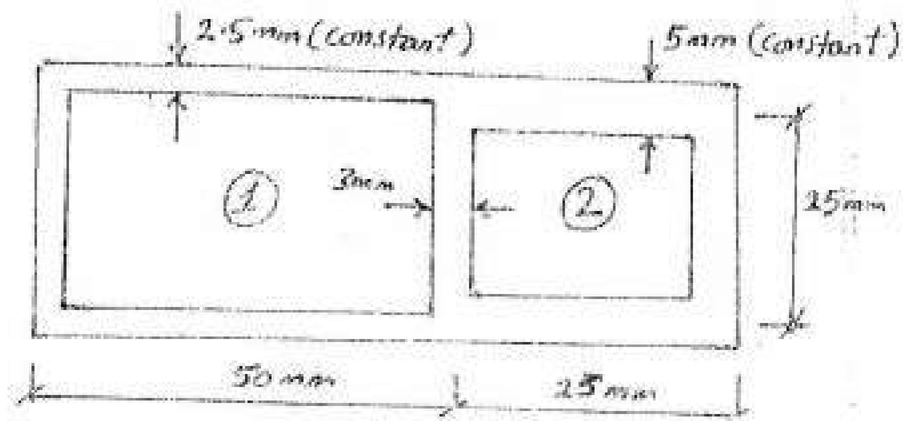
$$\boxed{= 2A_1 q_1 + 2A_2 q_2} \text{———} (1)$$

عندما تكون (t) ثخانة تساعد الحجم الحالي تماما
الآن

عبد الرحمن

Examples:

1. Consider the tube of two cells (shown here in). The torque in the tube is : $T = 200\,000$ N.mm , Also $G = 80\,000$ N/mm^2; calculate the shearing stresses in the walls of the section. Also calculate the twist for $l = 1000$ mm ?



Solution:

$$A_1 = 50 \times 25 = \underline{1250\ mm^2}$$
$$A_2 = 25 \times 25 = \underline{625\ mm^2}$$

First use the equilibrium of torques $(T = T_1 + T_2)$:

$$200\,000 = 2 \times 1250\, q_1 + 2 \times 625\, q_2 \quad ------ (1)$$

Next find the twist θ_1 of cell 1 :

$$\frac{G\theta_1}{l} = \frac{1}{2A_1}\left[q_1 \sum \frac{S_1}{t_1} + (q_1 - q_2) \cdot \frac{S_{1-2}}{t_{1-2}} \right]$$

$$\frac{G\theta}{l} = \frac{1}{2 \times 1250}\left[q_1\left(\frac{50}{2.5}+\frac{25}{2.5}+\frac{50}{2.5}\right) + (q_1 - q_2)\left(\frac{25}{3}\right)\right]$$

diaphragm

for cell 2:

$$\frac{G\theta_2}{l} = \frac{1}{2A_2}\left[q_2 \sum \frac{S_2}{t_2} - (q_1 - q_2)\frac{S_{1-2}}{t_{1-2}}\right]$$

$$= \frac{1}{2 \times 625}\left[q_2\left(\frac{25}{5}+\frac{25}{5}+\frac{25}{5}\right) - (q_1 - q_2)\frac{25}{3}\right]$$

use;

$$\frac{G\theta_1}{l} = \frac{G\theta_2}{l} \text{, then}$$

neg. sign

$$\frac{1}{2500}\left[q_1 \times 50 + \frac{25}{3}(q_1 - q_2)\right] = \frac{1}{1250}\left[q_2 \times 15 - \frac{25}{3}(q_1 - q_2)\right] \quad \text{------ (2)}$$

From (2), find $q_2 = \frac{15}{11} q_1$

substitute in (1).

$$200000 = 2500\, q_1 + 1250 \times \frac{15}{11} q_1$$

This gives; $q_1 = 47.567568$ N/mm

$q_2 = \frac{15}{11} \times 47.567568 = 64.864865$ N/mm

The shearing stresses will be: $\tau_1 = \frac{q_1}{t_1} = \frac{47.567568}{2.5}$

$= 19.027$ N/mm^2

السمك الأكبر يقع على [illegible]

diaframe [illegible] (5)

$$\tau_2 = \frac{q_2}{t_2} = \frac{64.864865}{5}$$

$$= 12.973\ N/mm^2\ \underline{MPa}$$

In the diaphragm : $\tau_{1-2} = \dfrac{q_1 - q_2}{t_{1-2}} = \dfrac{47.567568 - 64.86486}{3}$

shallow

$$= -5.766\ N/mm^2$$

If the sections are symmetric $\rightarrow q_1 = q_2 \Rightarrow \tau_{1-2} = 0$

Find $\theta = \theta_1$ or θ_2

$$\theta_1 = \frac{l}{2GA_1}\left[q_1 \sum \frac{S_1}{t_1} + (q_1 - q_2)\frac{S_{1-2}}{t_{1-2}}\right]$$

$$= \frac{1000}{2 * 80 * 10^3 * 1250}\left[47.6\left(\frac{50}{2.5} + \frac{25}{2.5} + \frac{50}{2.5}\right) + (47.6 - 64.9) * \frac{25}{3}\right]$$

$$= 0.011179\ \text{radian} * \frac{180}{\pi}$$

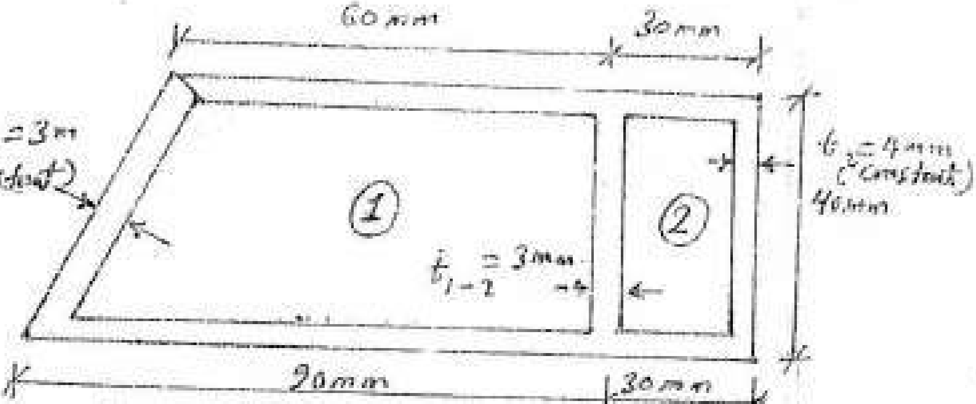
$$= 0.64^\circ$$

(o.k.)

Note: For symmetrical case $\rightarrow (q_1 = q_2) \Rightarrow \tau_{1-2} = 0$? bcs. the flow of shear is opposite ?

H.W.

The applied torque is 320 000 N.mm. Also $G = 80\,000\ N/mm^2$. Find the shearing stresses in the walls & also the angle of twist for $l = 1000$ mm ?



$A_1 = 3000\ mm^2 = \frac{60+90}{2} * 40 = 3000\ mm^2$

$A_2 = 30 \times 40 = 1200\ mm$

$T = T_1 + T_2 \Rightarrow 320 \times 10^3 = 6000\, q_1 + 2400\, q_2$ —— (1)

$$\frac{\theta_1 G}{l_1} = \frac{1}{2A_k}\left[q_1 \Sigma \frac{S_t}{t_1} + (q_1 - q_2)\Sigma \frac{S_{t-2}}{t_{1-2}}\right]$$

$$= \frac{1}{6000}\left[q_1 \Sigma\left[\frac{60}{3} + \frac{90}{3} + \frac{50}{3}\right] + (q_1 - q_2)\frac{40}{3}\right]$$

$$= \left[66.667\, q_1 + 13.33\, q_1 - 13.33\, q_2\right]$$

$$\frac{6000}{2400}\left[q_2\left[\frac{30}{4} + \frac{30}{4} + \frac{40}{4}\right] + 13.33\, q_2 - 13.33\, q_1\right]$$

$$\left[95.82\, q_2 - 33.33\, q_1\right] = 80\, q_1 - 13.33\, q_2$$

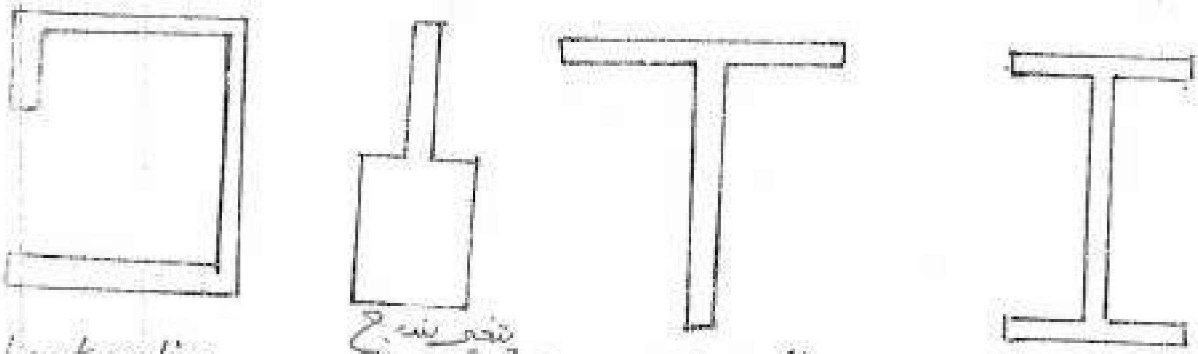
$113.33\, q_1 - 109.15\, q_2 = 0$ —— (2) Solve

$q_1 = 37.68$ $\quad \tau_1 = \frac{37.68}{3} = 12.55$ MPa

$q_2 = 39.12$ $\quad \tau_2 = \frac{39.12}{4} = 9.78$ MPa

$\boxed{\theta = 5.1 \times 10^{-3}\ \text{Rad}}$ $\boxed{0.297}$ $\quad \tau_{1-2} = \frac{37.66 - 39.12}{3} = -0.48$

Compound rectangular sections:



bent section — المقطع الملتوي; step varying section; T-section; I-section

For all these sections, use

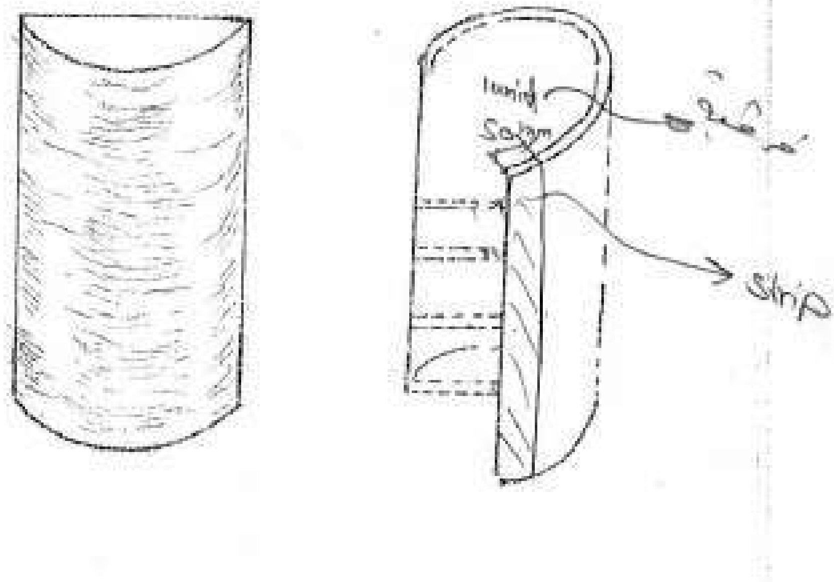
L-section

$$\boxed{J = \sum \frac{1}{3} t_i^3 b_i}$$

البعد الكلي

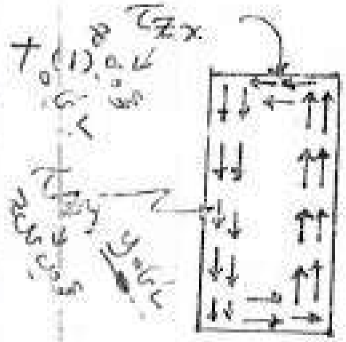
البعد العرضي

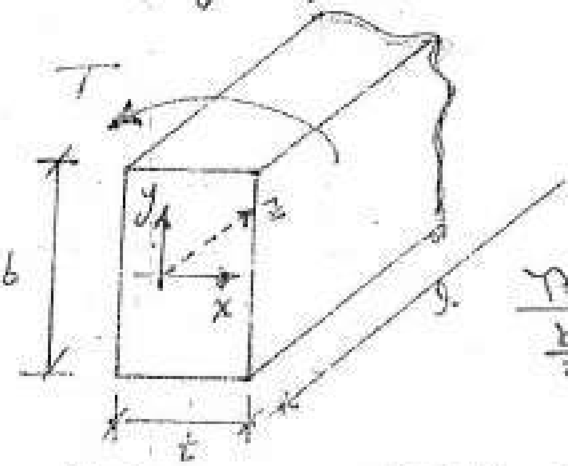
J: torsion constant of the section.



TORSION OF THIN RECTANGULAR SECTION

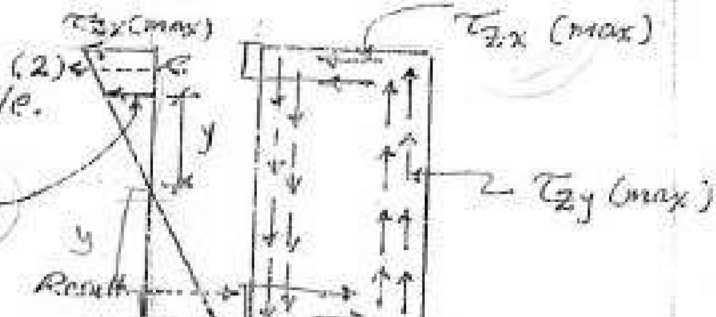
Consider a thin rectangular section (in x & y axes) under a torque T. The shearing stresses τ_{zx} must be parallel to the short edge & τ_{zy} to the long edge.(1)





$$\frac{\tau_{zx(max)}}{\frac{b}{2}} = \frac{\tau_{zx}}{y}$$

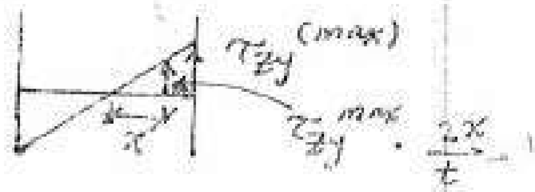
These shearing stresses, vary linearly from maximum at the edge to zero at the middle.



$$\tau_{zy} = \tau_{zx} \cdot \frac{2y}{b}$$

تشابه مثلثات

... solution, a good assumption ... made:



$$\frac{\tau_{zx}^{(max)}}{t} = \frac{\tau_{zy}^{(max)}}{b}$$

Thus; $\tau_{zx}^{(max)} = \frac{t}{b}\,\tau_{zy}^{(max)}$ (usually small)

$$\tau_{zy} \times \frac{t}{2} = \tau_{zx} \cdot \frac{b}{2}$$

ملخص

غير مهم

The torque T is resisted by both τ_{zy} & τ_{zx}. Using linear (or triangular) distribution, then:

$$T = \Big(\underbrace{\tfrac{1}{2}\,\tau_{zy}^{(max)}}_{stress} \cdot \underbrace{\tfrac{t}{2}\cdot b}_{area}\Big)\cdot\underbrace{\left(\tfrac{2}{3}\,t\right)}_{arm}$$

$$+ \Big(\underbrace{\tfrac{1}{2}\,\tau_{zx}^{(max)}}_{stress} \cdot \underbrace{\tfrac{b}{2}\cdot t}_{area}\Big)\cdot\underbrace{\left(\tfrac{2}{3}\,b\right)}_{arm}$$

$$= \frac{1}{6}\,\tau_{zy}^{(max)}\cdot t^2 b + \frac{1}{6}\,\tau_{zx}^{(max)}\cdot t b^2$$

$$= \frac{1}{6}\,\tau_{zy}^{(max)}\cdot t^2 b + \frac{1}{6}\left(\tau_{zy}^{(max)}\cdot\frac{t}{b}\right)\cdot t b^2$$

$$T = \frac{1}{3}\,\tau_{zy}^{(max)}\cdot t^2 b$$

Then:
$$\boxed{\tau_{zy}^{(max)} = \frac{3T}{t^2 b}}$$

Also;
$$\boxed{\tau_{zx}^{(max)} = \frac{3T}{t b^2}}$$

To find the torsion constant J or torsional rigidity GJ (or torsional stiffness $\frac{GJ}{\ell}$), use the method of real work.

$$\boxed{J = \sum \frac{1}{3}\, t_i^3\, b_i}$$

عبدالرحمن عبدالله محمد
عبدالرحمن عبدالله

Date: 8/2/2012

"Mechanics of Solids"

work by torque T = strain energy by shearing stresses.

Let θ be the angle of twist for a length "l";
then;

$$\boxed{W = \frac{1}{2} T \theta}$$

External work

compound

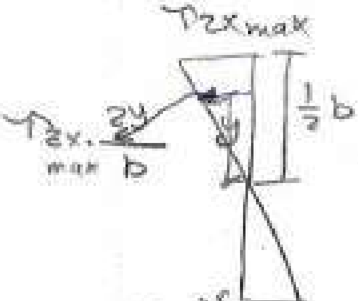
shear stress

Also, $$U = \int_{vol} \left(\frac{\tau_{zy}^2}{2G} + \frac{\tau_{zx}^2}{2G} \right) d(vol)$$

(as all other stresses are zero)

ثابت

$$= \int_{vol} \frac{\tau_{zy}^2}{2G} d(vol) + \int_{vol} \frac{\tau_{zx}^2}{2G} d(vol)$$



$$= 2 \int_0^{t/2} \frac{\left(\tau_{zy}^{(max)} \cdot \frac{2x}{t}\right)^2}{2G} \cdot \underbrace{l.b.dx}_{d(vol)}$$

$$+ 2 \int_0^{b/2} \frac{\left(\tau_{zx}^{(max.)} \cdot \frac{2y}{b}\right)^2}{2G} \cdot \underbrace{l.t.dy}_{d(vol)}$$

$$= \frac{4\left(\tau_{zy}^{(max)}\right)^2}{Gt^2} \cdot lb \int_0^{b/2} x^2 dx$$

$$+ \frac{4\left(\tau_{zx}^{(max)}\right)^2}{Gt^2} lt \int_0^{b/2} y^2 dy$$

98

$$= \frac{(\tau_{zy}^{(max)})^2}{G t^2}\, l b \cdot \frac{t^3}{6} + \frac{(\tau_{zx}^{(max)})^2}{G b^2}\, l t \cdot \frac{b^3}{6}$$

$$= \frac{\left(\frac{3T}{t^2 b}\right)^2 l b t}{6G} + \frac{\left(\frac{3T}{t b^2}\right)^2 l b t}{6G} = \frac{3}{2}\cdot\frac{T^2 l}{G t^3 b} + \frac{3}{2}\cdot\frac{T^2 l}{G t b^3}$$

$$\boxed{U = \frac{3}{2}\,\frac{T^2 l}{G t b}\left(\frac{1}{t^2} + \frac{1}{b^2}\right)}$$

write $W = U$, then

$$\frac{1}{2} T\theta = \frac{3}{2}\,\frac{T^2 l}{G t b}\left(\frac{1}{t^2} + \frac{1}{b^2}\right)$$

gives;

$$\theta = 3\,\frac{T l}{G t b}\left(\frac{1}{t^2} + \frac{1}{b^2}\right) = \frac{T l}{G(\frac{1}{3} t^3 b)}\left(1 + \frac{t^2}{b^2}\right)$$

The quantity $\frac{t^2}{b^2}$ is very small. Then;

[illegible]

$$\theta = \frac{T l}{G(\frac{1}{3} t^3 b)}$$

Defining J as, $\theta = \frac{T l}{G J}$

Then;

$$\boxed{J = \frac{1}{3} t^3 b}$$

, Compound rectangular sections use;

$$\boxed{J = \Sigma \frac{1}{3} t_i^3 b_i}$$

[illegible]

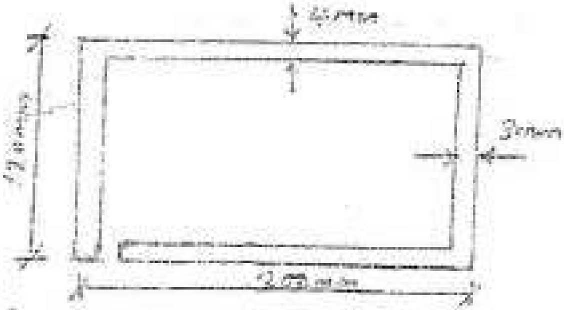
very thin with

Examples:

1. A tube of length $l = 800\,mm$ & a rectangular cell with a slit near one corner as shown here is under a torque $T = 2\ kN{\cdot}m$. Find the maximum shearing stresses & the angle of twist. Use $G = 80\ GPa$?

solution:

First find the torsion constant J for the multiple rectangles:



$$J = \sum \frac{1}{3} t_i^3 b_i$$

Thus:

$$J = \frac{1}{3}(3^3 * 120 * 2 + 4^3 * 200 * 2)$$

$$= 10693.333\ mm^4$$

$$= 0.010693 * 10^{-6}\ m^4$$

use;

$$\theta = \frac{T\,l}{GJ} \quad , Then; \quad \theta = \frac{2\,(kN{\cdot}m) * (0.8\,m)}{80 * 10^6\,(kN/m^2) * (0.01069 * 10^{-6}\,m^4)}$$

$$= \frac{1.6}{80 * 0.01069} = 1.871\ radian$$

(very big)

$$= 1.871 * \frac{180}{\pi} = \underline{\underline{107^\circ}} \quad ?$$

لاحظ القوة تبين ... وذلك بسبب الشق

* notice that the difference with respect to closed section (no slit)?

لاحظ دوران الماء في داخل القناة

* To find the maximum shearing stresses in these rectangles, first find the torque T_i in rectangle i, using:

$$\boxed{T_i = \frac{J_i}{J} \cdot T}$$

(As all rectangles have the same twist θ & (J_i/J) is the torque distribution factor).

This comes from; $T = T_1 + T_2 + \ldots$

$$\& \ \theta = \theta_1 = \theta_2 = \ldots$$

$$\text{Then} \quad \frac{T l}{GJ} = \frac{T_1 l}{GJ_1} = \frac{T_2 l}{GJ_2}$$

Thus;

$$T_1 = \frac{J_1}{J} \cdot T$$

$$T_2 = \frac{J_2}{J} \cdot T$$

$$\vdots$$

For rectangles 3mm * 120mm : $J_1 = \frac{1}{3} * 3^3 * 120$

$$= 1080 \ mm^4$$

Then;

$$T_1 = \frac{1080}{10693} * 2 * 10^6 = 202.0 * 10^3 \ N \cdot mm$$

Use $\left(\tau_{zy}^{(max.)}\right)_1 = \frac{3 T_i}{t_i^2 b_i} = \frac{3 * 202.0 * 10^3}{3^2 * 120} = \underline{\underline{561 \ N/mm^2}}$

For rectangles (4mm * 200mm) : $J_2 = \frac{1}{3} * 4^3 * 200$

$$= 4266.7 \ mm^4$$

Thus; $T_2 = \frac{J_2}{J} \cdot T$

$$= \frac{4266.7}{10693} * (2*10^6)$$

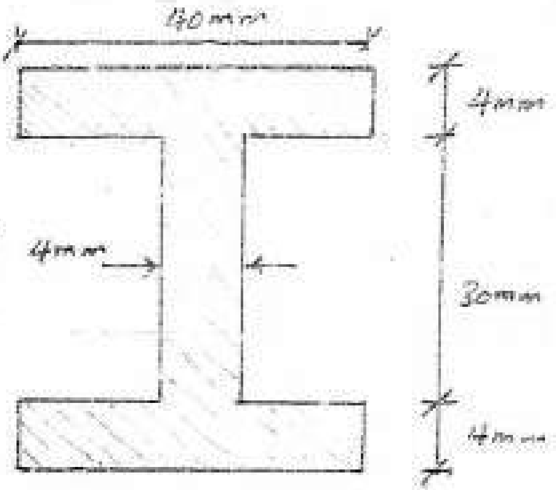
$$= 797.942 * 10^3 \ N.mm$$

Then;

$$(\tau_{zy}^{(max)})_2 = \frac{3T_2}{t_2^2 b_2} = \frac{3*797.942*10^3}{4^2 * 200} = \underline{\underline{748}} \ N/mm^2$$

high value

Ex. 2 An I-section under torque, find the angle of twist (per 1m) & the maximum shearing stresses? G = 80 GPa (for steel)?



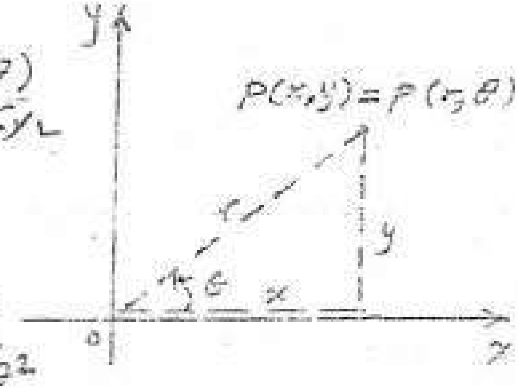
TWO-DIMENSIONAL PROBLEMS IN ELASTICITY IN POLAR COORDINATES

Polar Coordinates: A point "p" in x & y coordinates can also be described in polar coordinates (r & θ). There are relations between these two types of coordinates:

$$r = \sqrt{x^2+y^2} \quad ; \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Also; $\boxed{x = r\cos\theta}$, $\boxed{y = r\sin\theta}$

using these relations, then all equations in x & y can be converted into r & θ.



* (all equs. can be convert from x & y → to r & θ)

$r\cos\theta \leftarrow x$ [illegible]

$r\sin\theta \leftarrow y$

For example:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Thus, in x & y, the basic (or governing) equation for 2-dimensional stresses is;

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

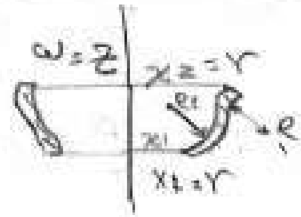
where $\phi = \phi(x,y)$ — (ϕ function of x & y)

or $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0 \qquad \nabla^2\cdot\nabla^2\phi = 0$

In polar coordinates (r & θ), the basic (or governing) equation becomes:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\boxed{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)=0$$

problems in polar coordinates are in two types:



يعتمد على r بدون θ

1. Axisymmetric problems: المحور الذي يتم تدوير عليه

Here, all quantities depend on the radial distance r only (not on the angle θ). Thus here;

$$\nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}$$

تم حذفة لكونه عرضي

لحسابات

2. Asymmetric problems:

Here, all quantities depend on both r & θ.

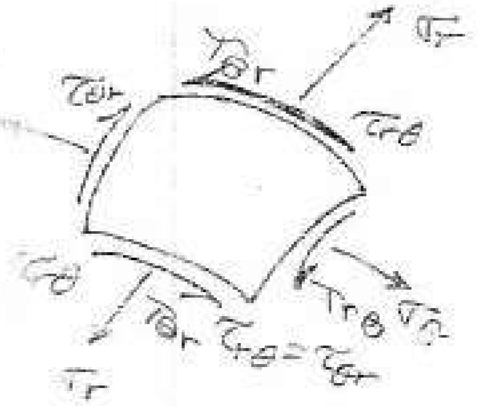
$$\nabla^2\phi\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)=0$$

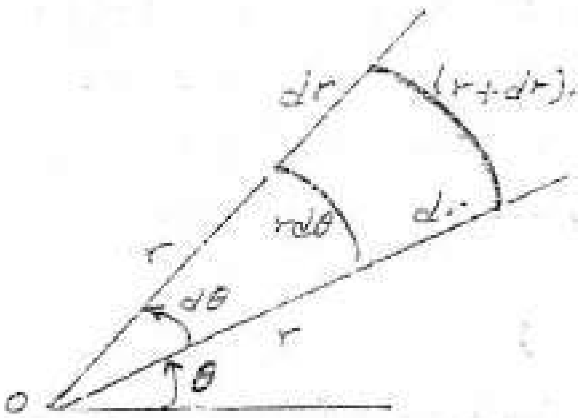
~~$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = \frac{-2pr^2}{2\pi} \sin 2\theta$~~ =

$$\sigma_\theta=\frac{\partial^2\phi}{\partial r^2}=-\frac{P}{\pi}\left(\theta-\frac{1}{2}\sin 2\theta\right)$$

$$\sigma_r=\frac{-P}{\pi}\left(\theta+\frac{1}{2}\sin 2\theta\right)$$

Stresses in polar coordinates:





Take a small element (of area $dr.rd\theta$). The stresses on the sides of this element are:

1. The radial normal stress σ_r.
2. " tangential " " σ_θ.
3. " shearing stresses $\tau_{r\theta} = \tau_{\theta r}$

الإجهاد العمودي الشعاعي
الإجهاد المماسي (عمودي) ولكن نعرف محل
تأثيره سطح مماسي

Notice that the two normal stresses "σ_θ" are not in one line. Similarly the two shearing stresses "$\tau_{\theta r}$" are not parallel

Equation of equilibrium of stresses in polar coordinates:

بحث 24%

when the stresses vary as r & θ vary, the two equations of equilibrium are:

the drive

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{r\theta}}{r\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad \text{- - - - (1)} \quad \text{(in radial r-direction)}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \sigma_\theta}{r\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0 \quad \text{- - - (2)} \quad \text{(in tangential } \theta\text{-dir.)}$$

There are 3 unknowns (σ_r, σ_θ & $\tau_{r\theta} = \tau_{\theta r}$), but only 2 equations of equilibrium. Then the problem is statically indeterminate. One more equation from the deformations are needed.

Strains in polar coordinates:

Let:

u be the displacement in r-direction.

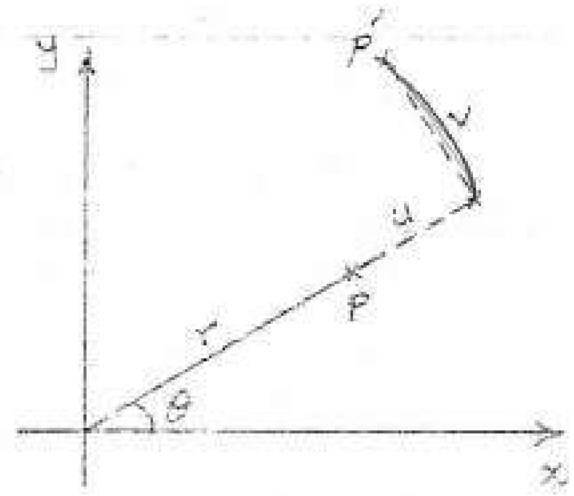
v be the displacement in θ-direction.

The strains are:

1. Normal strain in r-direction; $\epsilon_r = \frac{\partial u}{\partial r}$
2. Normal strain in θ-direction; $\epsilon_\theta = \frac{\partial v}{r\partial \theta} + \frac{u}{r}$
3. Shearing strain rθ-plane

$$\gamma_{r\theta} = \gamma_{\theta r} = \frac{\partial v}{\partial r} + \frac{\partial u}{r\partial \theta} - \frac{v}{r}$$

see the Figure.



Stress _ strain relations:

Use Hooke's law (linear elasticity) then,

$$\boxed{\epsilon_r = \frac{\sigma_r}{E} - \nu \frac{\sigma_\theta}{E}}$$

$$\boxed{\epsilon_\theta = \frac{\sigma_\theta}{E} - \nu \frac{\sigma_r}{E}}$$

$$\boxed{\gamma_{r\theta} = \frac{\tau_{r\theta}}{G}} \quad \& \quad \boxed{G = \frac{E}{2(1+\nu)}}$$

or,

$$\boxed{\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta)}$$

$$\boxed{\sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r)}$$

$$\boxed{\tau_{r\theta} = G\gamma_{r\theta}}$$

Basic (or governing) equation:

As mentioned before, the basic or governing equation can be obtained either from $\nabla^4\phi = 0$ in x & y, or from the equations of equilibrium

with stress-strain relations. Define a stress function $\phi = \phi(r, \theta)$. The stresses are given by:

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial \phi}{r\partial \theta}\right) \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial \phi}{\partial \theta}\right)$$

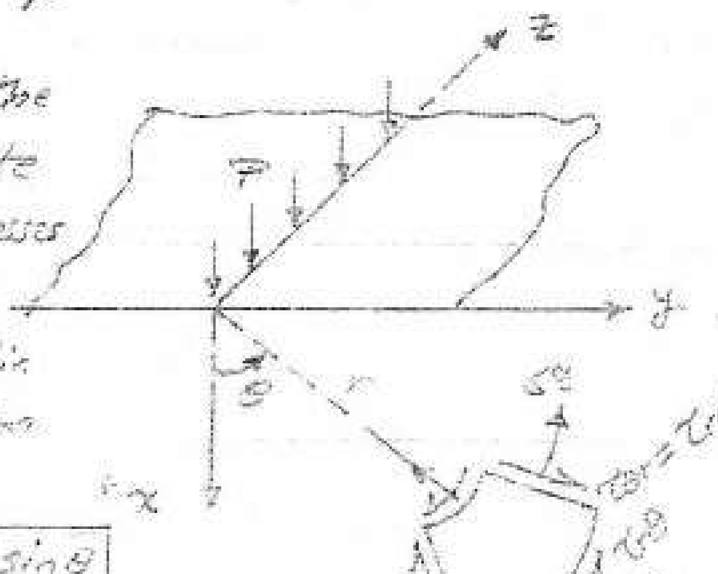
Thus, if $\phi = \phi(r, \theta)$ is known, then all stresses can be calculated.

Applications:

The most important applications is the stresses inside a semi-infinite medium under a line load (Boussinesque solution).

Let P (per unit width) be the line load on a semi-infinite medium. To determine the stresses on an element ($dr\, r d\theta$) at distance "r" & angle of dir. θ, the following stress function is tried:

$$\boxed{\phi(r, \theta) = C\, r\, \theta \sin\theta}$$



$\phi(r,\theta) = c\, r\, \theta \sin\theta$ → where c is a constant (to be determined). This stress function ([illegible]) satisfies the basic (or governing) equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{r^2\partial\theta^2}\right) = 0$$

Thus, this is a valid stress function
next find the stresses:

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\therefore \sigma_r = \frac{c}{r}\,\theta\sin\theta + \frac{cr}{r^2}\frac{\partial}{\partial\theta}\left\{\theta\cos\theta + \sin\theta\right\}$$

$$= \frac{c}{r}\,\theta\sin\theta + \frac{c}{r}\left(-\theta\sin\theta + \cos\theta + \cos\theta\right)$$

$$= \frac{2c}{r}\cos\theta$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = \frac{\partial}{\partial r}\left(c\,\theta\sin\theta\right) = 0$$

$$\tau_{r\theta} = \frac{-\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

$$= -\frac{\partial}{\partial r}\left\{\frac{cr}{r}\left(\theta\cos\theta + \sin\theta\right)\right\} = 0$$

Thus only radial normal stress exists.

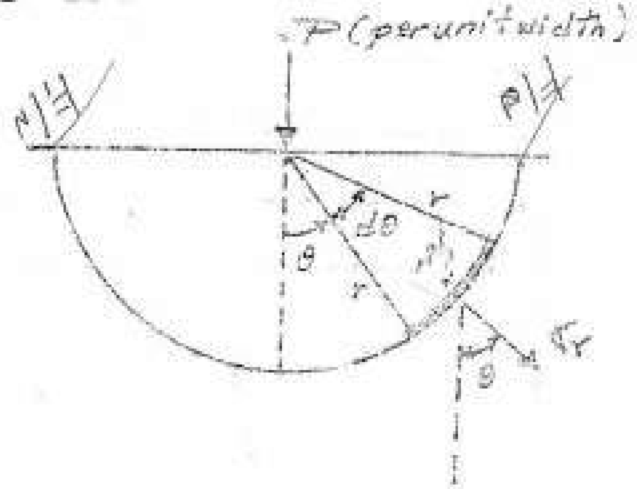
Next find "C". Take a $\frac{1}{2}$-cylinder of any radius "r":

Use vertical equilibrium:

$$P.\underset{\substack{\uparrow \\ (1\text{-unit})}}{1} + \int_{-\frac{\pi}{2}}^{+\pi/2} (\underbrace{\sigma_r}_{\text{stress}} . \underbrace{r\,d\theta . 1}_{\text{area}}) . \underbrace{\cos\theta}_{\substack{\text{vert.} \\ \text{compon.}}} = 0$$

Use $\boxed{\sigma_r = \frac{2c}{r}\cos\theta}$, then

$$P + \int_{-\pi/2}^{+\pi/2} \left(\frac{2c}{r}\cos\theta\right) . r\,d\theta . \cos\theta = 0$$



$$P + \int_{-\pi/2}^{+\pi/2} 2c . \cos^2\theta . d\theta = 0$$

$$P + 2c\int_{-\pi/2}^{+\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = 0$$

$$P + c\left/ \theta - \frac{\sin 2\theta}{2} \right/_{-\pi/2}^{+\pi/2} = 0$$

Thus; $P + c.\pi = 0$, this gives $\boxed{c = -\frac{P}{\pi}}$

Thus;

$\boxed{\phi = -\frac{P}{\pi}\, r\theta \sin\theta}$, $\boxed{\sigma_r = -\frac{2P}{\pi} . \frac{\cos\theta}{r}}$

$\boxed{\sigma_\theta = 0 \quad \& \quad \tau_{r\theta} = 0}$

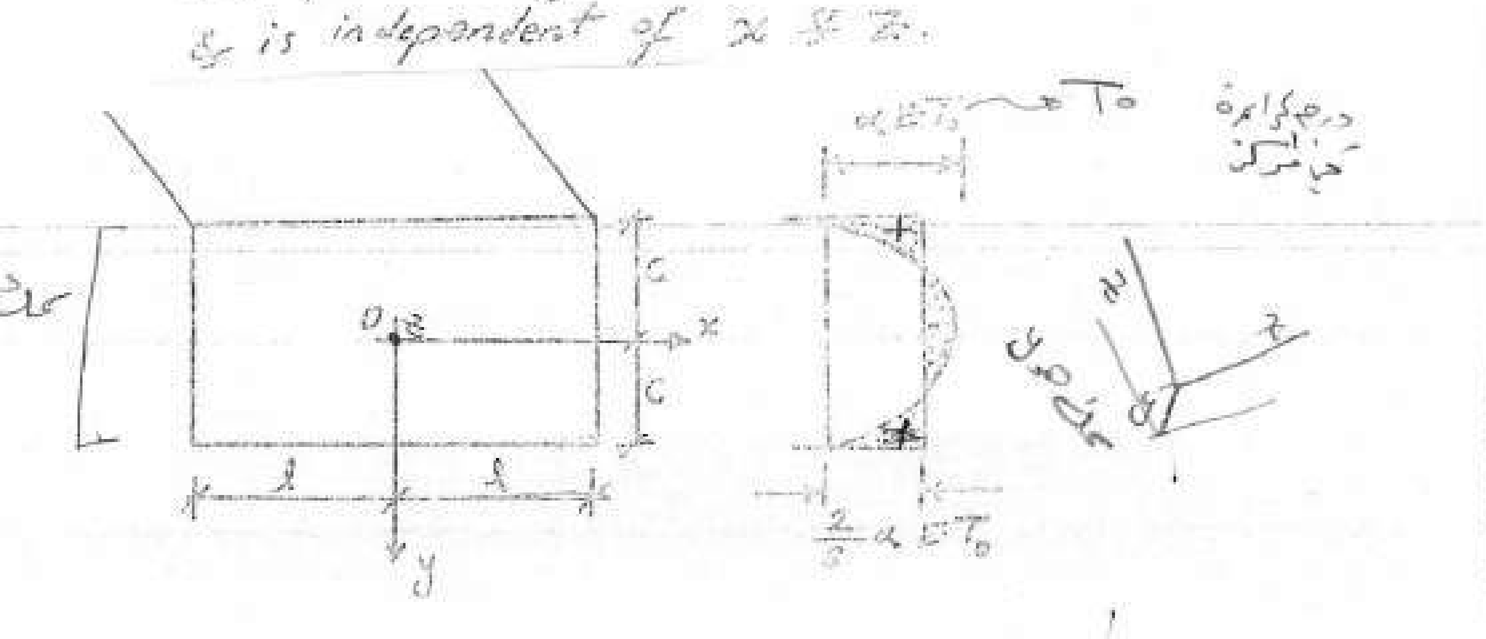
Thermal stresses

The simplest cases of thermal stress distribution
(Method of strain suppression): طريقة اشتقاق الاجهادات

One of the causes of stress in a body is nonuniform heating, with rising temperature the elements of a body expand. Such an expansion generally cannot proceed freely in a continous body, & stresses due to the heating are set up.

Fracture of glass when a surface is rapidly heated is attributable to such stress. Fatigue failure can occur as a result of temperature fluctuations. The consequences of such thermal stress are important in many aspects of engineering design, as in turbines, jet engines, & nuclear reactors.

The simpler problems of thermal stress can easily be reduced to problems of boundary force of types already considered. As a first example let us consider a thin rectangular plate of uniform thickness in which the temperature "T" is an even function of "y", as shown in fig. below, & is independent of x & z.



The longitudinal thermal expansion αT will be entirely suppressed by applying to each element of the plate longitudinal stress,

$$\sigma_x' = -\alpha T E \quad \text{- - - -} \quad (1)$$

زيادة $\quad \sigma_x = +$

which is compressive when T is positive. Since the plate is free to expand laterally, the application of the stress (1) will not produce stress in the lateral directions. To maintain the stress (1) throughout the plate, it will be necessary to distribute compressive forces of the magnitude (1) at the ends of the plate only. These compressive forces will completely suppress any expansion of the plate in the direction of the x-axis due to the temperature rise T. To get the termal stress in the plate, which is free from external forces, we have to superpose on the stress (1) the stresses produced in the plate by tensile forces of intensity $\alpha T E$ distributed at the ends. These forces have the resultant;

$$\int_{-c}^{+c} \alpha T E \cdot dy$$

& at a sufficient distance from the ends they will produce approximately uniformly distributed tensile stress of the magnitude;

$$\frac{1}{2c} \int_{-c}^{+c} \alpha T E \cdot dy$$

So that the thermal stress in the plate with free ends, at a sufficient distance from the ends, will be,

$$\boxed{\sigma_x = \frac{1}{2c} \int_{-c}^{+c} \alpha T E \cdot dy - \alpha T E} \quad \text{- - -} \quad (2)$$

شرح الأول ، الحرارة توزيع بشكل parabolic

Assuming, for example, that the temperature is distributed parabolically & is given by the equation:

درجة الحرارة $(T) = T_0 \left(1 - \frac{y^2}{c^2}\right)$ when $y = 0$ mid
معادلة
$T = T_0$ thic.

(2) we get, from eq. (2),

$$\sigma_x = \frac{2}{3} \alpha T_0 E - \alpha T_0 E \left(1 - \frac{y^2}{c^2}\right) \quad \text{---} (3)$$

توزيع الإجهادات (1)

This stress distribution is shown in fig. above.

قريب النهاية الشكل

Near the ends the stress distribution produced by the tensile forces is not uniform & must be calculated by methods suitable for end effects, such as those explained, Castigliano's theorem & stress distribution symmetrical about an axis (polar coord.).

بالإمكان توقع الإجهادات الناتجة عن الشد عن طريق polar or Castigliano

If the temperature T is not symmetrical with respect to the x axis, we begin again with compressive stress (1) suppressing the strain ϵ_x. In the nonsymmetrical cases this stress gives rise not only to a resultant force $\left(-\int_{-c}^{+c} \alpha E T\, dy\right)$ but also to a resultant couple $\boxed{-\int_{-c}^{+c} \alpha E T y\, dy}$ & in order to satisfy the conditions of equilibrium, we must superpose on the compressive stresses (1) a uniform tension, determined as before, & bending stresses $\boxed{\sigma''_x = \sigma y / c}$ determined from the condition that the moment of the forces distributed over a cross section must be zero. Then

Couple بسبب (y)

شرط التوازن
تحقق satisfy
the B.C. Equilibrium

$$\underbrace{\int_{-c}^{+c} \frac{\sigma y^2\, dy}{c}}_{\text{Comp}} - \underbrace{\int_{-c}^{+c} \alpha E T y\, dy}_{\text{Ten}} = 0$$

from which; $\dfrac{\sigma}{c} = \dfrac{3}{2c^3}\int_{-c}^{+c} \alpha . E . T . y . dy$, $\sigma_x'' = \dfrac{3y}{2c^3}\int_{-c}^{+c} \alpha E T y \, dy$

$F = (k) . \Delta\delta$

$P = \dfrac{EA}{L} . \Delta L$

Then the total stress is;

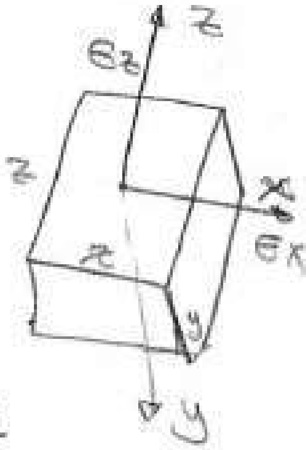
$$\boxed{\sigma_x = -\alpha E T + \frac{1}{2c}\int_{-c}^{+c} \alpha E T \, dy + \frac{3y}{2c^3}\int_{-c}^{+c} \alpha E T y \, dy \quad \text{---- (4)}}$$

In this discussion it was assumed that the plate was thin in the z-direction(t). suppose now that the dimension in the z-direction is large. We have then a plate with the xz plane as its middle plane & a thickness 2c. Let the temperature T be, as before, independent of x & z, & so a function of y only. The free thermal expansion of an element of the plate in the x & z directions will be completely suppressed by applying stresses σ_x, σ_z obtained from eqns;

$$\epsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right)$$

$$\epsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$$

$$\epsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right)$$



By putting, $\boxed{\epsilon_z = \epsilon_x = -\alpha T}$, $\sigma_y = 0$

These equations then give;

[illegible] = 0

$\sigma_y = 0$

$$\sigma_x = \sigma_z = -\frac{\alpha E T}{1-\nu} \quad \text{---- (5)}$$

Finally arrive at the equation:

$$\boxed{\sigma_x = \sigma_z = -\frac{\alpha T E}{1-\nu} + \frac{1}{2c(1-\nu)}\int_{-c}^{+c} \alpha T E . dy + \frac{3y}{2c^3(1-\nu)}\int_{-c}^{+c} \alpha T E y \, dy \quad \text{---- (6)}}$$

which is analogous to Eq. ④. From eq. ⑥ we can easily calculate thermal stresses in a plate, if the distribution of temperature T over the thickness of the plate is known.

<u>Example</u>: Consider a plate which has initially a uniform temperature T_0 & which is being cooled down by maintaining the surfaces $y = \pm c$ at a constant temperature T_1. By Fourier's theory the distribution of temperature at any instant t is;

odd function and angle factor

$$T = T_1 + \frac{4}{\pi}(T_0 - T_1)\left(e^{-P_1 t}\cos\frac{\pi y}{2c} - \frac{1}{3}e^{-P_3 t}\cos\frac{3\pi y}{2c} + \cdots\right) \quad \cdots ⑦$$

in which $P_1, P_3 = 3^2 P_1, \ldots, P_n = n^2 P_1 \ldots$ are certain constants. substituting in eq. ⑥, we find;

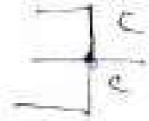
$$\boxed{\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1-\nu)}\Big[e^{-P_1 t}\left(\frac{2}{\pi} - \cos\frac{\pi y}{2c}\right)}$$
$$+ \frac{1}{3}e^{-P_3 t}\left(\frac{2}{3\pi} + \cos\frac{3\pi y}{2c}\right) + \frac{1}{5}e^{-P_5 t}\left(\frac{2}{5\pi} - \cos\frac{5\pi y}{2c}\right) + \cdots\Big) \quad \cdots ⑧$$

• After a moderate time the first term acquires dominant importance, & we can assume;

<u>Rep</u>

$$\boxed{\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1-\nu)}e^{-P_1 t}\left(\frac{2}{\pi} - \cos\frac{\pi y}{2c}\right)}$$

For $y = \pm c$ we have tensile stresses ;



$$\sigma_x = \sigma_z = + \frac{4\alpha E (T_0 - T_1)}{\pi (1-\nu)} e^{-p_1 t} \cdot \frac{2}{\pi}$$

At the middle plane $y=0$ we obtain compressive stresses ,

$$\sigma_x = \sigma_z = - \frac{4\alpha E (T_0 - T_1)}{\pi (1-\nu)} e^{-p_1 t} \left(1 - \frac{2}{\pi}\right)$$

The point with zero stresses are obtained from the equation ,

$$\frac{2}{\pi} - \cos \frac{\pi y}{2c} = 0$$

$\cos 0 = 1$

from which $y = \pm 0.560C$

Thermal Stresses

EX: Consider a plate which has initially a uniform temperature T_0 which is being heated up by maintaining the surfaces $y = \pm c$ at a constant temperature T_1. By Fourier's Theory the distribution of temperature at any instant t is ;

$$T = T_1 + \frac{4}{\pi}(T_0 - T_1)\left(e^{-P_1 t}\cos\frac{\pi y}{2c} - \frac{1}{3}e^{-P_3 t}\cos\frac{3\pi y}{2c} + \cdots\right)$$

* Find the tensile & compressive stresses? And then find the point of zero stress? After a moderate time; use the first term which is dominant?

where ; $$\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1-\nu)} e^{-P_1 t}\cdot\left(\frac{2}{\pi} - \cos\frac{\pi y}{2c}\right)$$

$\alpha = 1.25 * 10^{-5}$

$E = 200 * 10^3$ MPa (steel, $f_y = 375$ MPa)