بسم الله الرحمن الرحيم

Engineering Analysis and Numerical Methods

Stage: Third

Civil Engineering Department

Text Book

Advanced Engineering Mathematics

By

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References

- Advanced Engineering Mathematics by Kreyszig
- Differential Equation by Iyengar
- Advanced Mathematics by Agarwal , et .al
- Integral Calculus and Differential Equations by Chatterjee

Syllabus

❖Ordinary Differential Equations of First order	(16 hrs)
❖Linear Differential Equations with Constant Coefficient	(12 hrs)
❖Simultaneous Linear Differential Equations	(12 hrs)
❖Numerical Solutions of Ordinary Differential Equations	(8 hrs)
❖Finite Differences	(4 hrs)
❖Interpolation	(4 hrs)
❖Numerical Differentiation	(8 hrs)
❖Numerical Integration and Computer Application	(4 hrs)
❖Fourier Series	(16 hrs)
❖Partial Differential Equation and Boundary Value problems	(12 hrs)
❖Numerical Solution for Partial Differential Equations	(8 hrs)
❖Matrices and its Applications	(16 hrs)

Ordinary Differential Equations of First Order

*******Definitions:***

Differential equation (DE) : An equation involves one or more derivatives or differentials.

*******Type:*** Ordinary or Partial:

Ordinary derivatives occur when the dependent variable "y" is a function of one independent variable "x"; $y = f(x)$

Partial derivatives occur when the dependent variable "y" is a function of two or more independent variables ; i.e.

$$y = f(x, t)$$

*******Order*** : (highest derivative)

*******Degree:*** (power of highest derivative)

Example (1)

$$x^2\bar{\bar{y}} + \bar{y} + (x^2 - 4)y = 0$$

Ordinary, Order 2, Degree 1

$$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2+1} = e^x$$

Ordinary, Order 3, Degree 2

Example (2)

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

Partial, Order 4, Degree 1

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Partial, Order 2, Degree 1

Linear and non-Linear differential equations

If a differential equation is of first degree in the dependent variable y and its derivatives (consequently , there cannot be any term involving the product of y and its derivatives) then it is called a linear differential equation otherwise it is non -linear .

OR

A linear differential equation (of y = f(x)) is of the form :

$$a_o y + a_1 \bar{y} + a_2 \bar{\bar{y}} + a_3 y^{\equiv} + \cdots \ldots \ldots \ldots + a_n y^{(n)} = b \tag{1}$$

Or

$$p_o y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_{n-1}\bar{y} + p_n y = r(x)$$

A non linear D.E. ,cannot be put in form (1) .

Examples:

$$\bar{\bar{y}} + 4x\bar{y} + 2y = \cos x$$ linear

$$\bar{\bar{y}} + 4y\bar{y} + 2y = \cos x$$ non linear because ($y\bar{y}$)

$$\bar{\bar{y}} + \sin y = 0$$ non linear because ($siny$)

A solution of D.E.; is a relation between the dependent and independent variables, and it satisfies the equation identically:

$$y = a\cos x + b\sin x$$

is a general solution of:

$$\frac{d^2y}{dx^2} + y = 0$$

The general solution of D.E. of n^{th} order, is one contains n *essential constants* (parameters). By essential we mean that the n constants cannot be replaced by a smaller number.

For example,

$$a\cos^2 x + b\sin^2 x + c\cos 2x$$

contains 3 constants and can be reduced

$$a\cos^2 x + b\sin^2 x + c(cos^2 x - sin^2 x)$$

$$= (a+b)cos^2 x + (b-c)sin^2 x$$

$$= d\ cos^2 x\ + e\ sin^2 x$$

Where $d = a + b$ and $e = b - c$

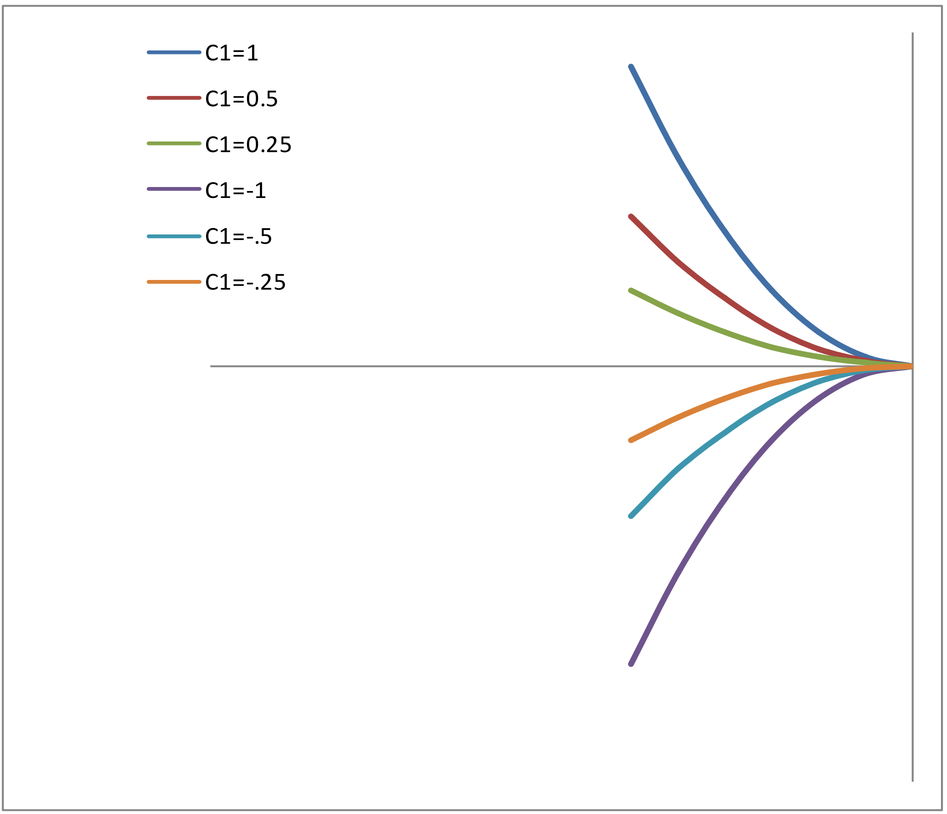
However, there are equations such that;

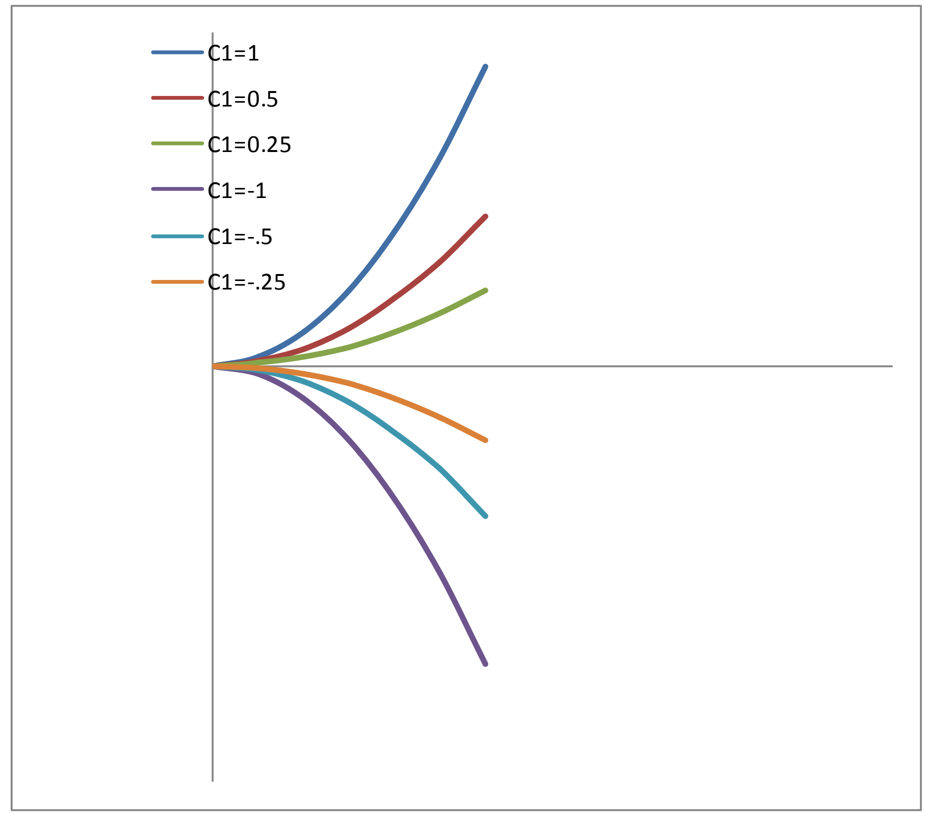
$$\left|\frac{dy}{dx}\right| + |y| = 0$$ (which has only the single solution *y=0*)

$$\left|\frac{dy}{dx}\right| + 1 = 0$$ (which has no solutions at all)

Also, there are differential equations which posses solutions, containing more essential parameters than the order of the equation.

$$y = \begin{cases} c_1 x^2 & (x \le 0) \\ c_2 x^2 & (x \ge 0) \end{cases}$$





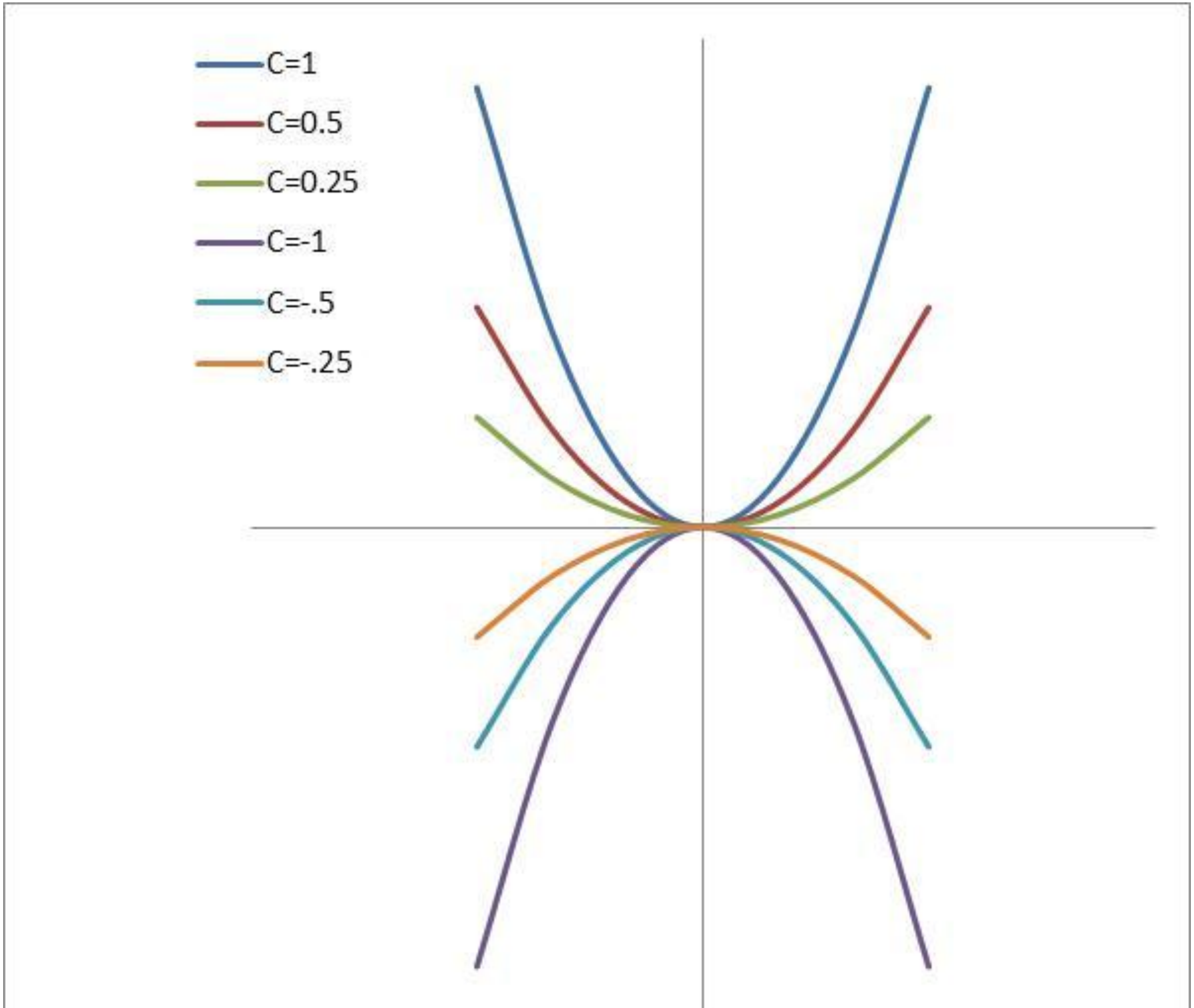
Both can be pieced together to give a D.E.

$$\bar{y} = \frac{2y}{x} \; for \;\; all \;\; values \;\; of \;\; x$$

$$c_1 = \frac{y}{x^2}, \;\; \bar{y} = 2c_1 x = \frac{2y}{x}$$

$$c_2 = \frac{y}{x^2}, \;\; \bar{y} = 2c_2 x = \frac{2y}{x}$$

$$Also \quad \bar{y} = \frac{2y}{x} \;\; for \quad y = cx^2$$


C=1
C=0.5
C=0.25
C=-1
C=-.5
C=-.25

Any solution found from the general solution by assigning particular values to the constants is called *a particular solution.*

$$y = a\cos x + b\sin x$$ <u>*general solution*</u>

$$a = 1 \quad and \quad b = 0$$

$$y = \cos x$$ <u>*particular solution*</u>

Solution which cannot be obtained from any general solution by assigning specific values to the constants are called ***singular solution***.

If a general solution has the property that every solution of the differential equation can be obtained from it by assigning suitable values to its arbitrary constants, it is said to be a ***complete solution.***

Example: Show $y = ae^{-x} + be^{2x}$ is a solution of

$y'' - y' - 2y = 0$ for all values of a and b.

Solution:

$$y'' - y' - 2y = (ae^{-x} + 4be^{2x}) - (-ae^{-x} + 2be^{2x})$$
$$- 2(ae^{-x} + be^{2x})$$

$$=(e^{-x}+e^{-x}-2e^{-x})a+(4e^{2x}-2e^{2x}-2e^{2x})b$$

$$=0a+0b=0$$ o.k.

<u>Note:</u>

$$y_1=ae^{-x} \quad or \quad y_2=be^{2x}$$

Satisfies

$$yy''-(y')^2=0$$

$$y_1y_1''-(y_1')^2=0$$

$$y_2y_2''-(y_2')^2=0$$

But;

$$y = y_1 + y_2 = ae^{-x} + be^{2x}$$

<u>Is not solution of</u>

$$yy'' - (y')^2 = 0$$

Since $yy'' - (y')^2 = 0$ Is not linear, while;

$$y'' - y' - 2y = 0$$

Is linear

__Example__: Given $y = ae^x + b\cos x$ (1)

Find second-order differential equation.

__Solution:__

If the given function has n constants, differentiate n[th] times and then eliminate the constants.

$$y' = ae^x - b\sin x \qquad (2)$$

$$y'' = ae^x - b\cos x \qquad (3)$$

By adding and subtract equations (1) and (3)

$$y + y'' = 2ae^{x} \Longrightarrow a = \frac{y + y''}{2e^{x}}$$

$$y - y'' = 2b\cos x \Longrightarrow b = \frac{y - y''}{2\cos x}$$

Substitute a and b into Eq. (2)

$$y' = \frac{y + y''}{2e^{x}}e^{x} - \frac{y - y''}{2\cos x}\sin x$$

$$2y' = y + y'' - y\tan x + y''\tan x$$

$$\boxed{(1+\tan x)y'' - 2y' + (1-\tan x)y = 0}$$

$$ae^{x}+b\cos x-y=0 \qquad (1)$$

$$ae^{x}-b\sin x-y'=0 \qquad (2)$$

$$ae^{x}-b\cos x-y''=0 \qquad (3)$$

$$\begin{vmatrix} 1 & +\cos x & -y \\ 1 & -\sin x & -y' \\ 1 & -\cos x & -y'' \end{vmatrix}=0$$

$$(y''\sin x-y'\cos x)-(-y''\cos x-y\cos x)$$

$$+(-y'\cos x-y\sin x)=0$$

$$y''(\cos x + \sin x) - 2y' \cos x - y(\cos x - \sin x) = 0$$

Divide by *cos x* getting:

$$y'' + y'' \tan x - 2y' + y - y \tan x = 0$$

$$(1 + \tan x) y'' - 2y' + (1 - \tan x) y = 0$$

This is only second-order D.E., but there are other D.E.

Differentiate Eq. (3) $y'' = ae^x - b\cos x$ twice more,

$$\boxed{y^{iv} = ae^x + b\cos x}$$

$$\boxed{y^{iv} = y}$$

and since it is a 4th order D.E. we expect the solution to contain 4 constants and we can show that:

$$\boxed{y = ae^x + b\cos x + ce^{-x} + d\sin x}$$

Satisfied $y^{iv} = y$ for all values of a, b, c and d.

Separable First-Order D.E.

Often a first order D.E. can be reduce to

$$f(x)dx = g(y)dy \qquad (1)$$

And such an equation is said to be ***separable;***

The general solution is:

$$\int f(x)dx = \int g(y)dy + C \qquad (2)$$

Other forms:

$$f(x)G(y)dx = F(x)g(y)dy \qquad (3)$$

$$\frac{dy}{dx} = M(x)N(y) \qquad (4)$$

Solution of Eq. (3) is:

$$\int \frac{f(x)}{F(x)}dx = \int \frac{g(y)}{G(y)}dy + C$$

And the solution of Eq. (4) is:

$$\int \frac{dy}{N(y)} = \int M(x)dx + C$$

Example: Solve $xy' = y + 1$

Solution:

$$\frac{y'}{y+1} = \frac{1}{x}$$

$$\frac{dy}{y+1} = \frac{dx}{x}$$

$$\ln(y+1) = \ln(x) + c$$

$$= \ln(x) + \ln(a)$$

$$y + 1 = xa$$

Particular solution curve means one-member

For all values of a straight lines pass through (0,-1)

b.) Find the ***orthogonal trajectories curves.***

Slope of straight line $=\dfrac{dy}{dx}=y'=a$

Or $$y'=\frac{y+1}{x}$$

Hence, the slope of the orthogonal trajectories

$$y'=-\frac{x}{y+1}$$

$$y' = -\frac{x}{y+1}$$

$$(y+1)dy = -xdx$$

$$\frac{y^2}{2} + y = -\frac{x^2}{2} + c$$

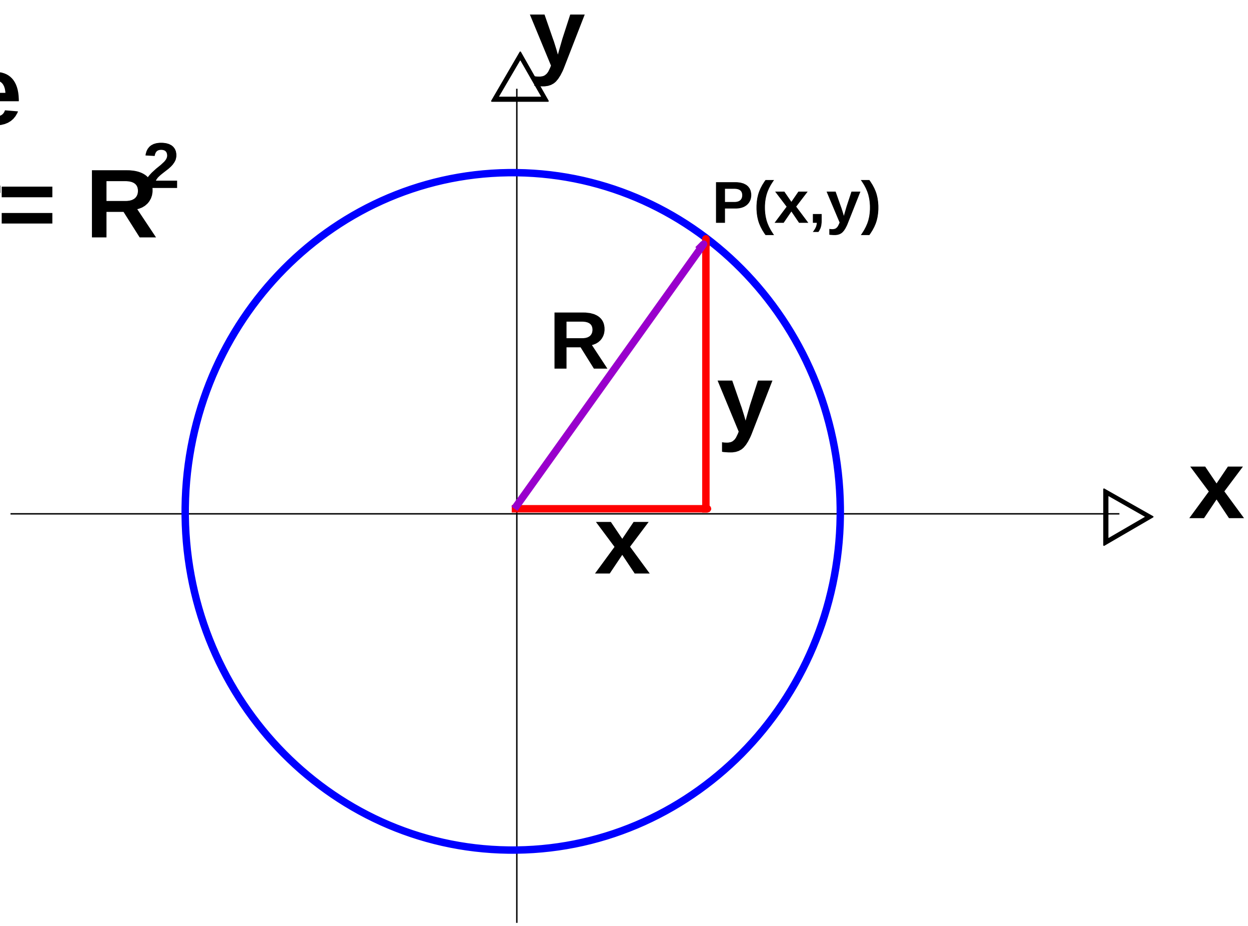
$$x^2 + y^2 + 2y = 2c$$

$$x^2 + (y^2 + 2y + 1) = 2c + 1$$

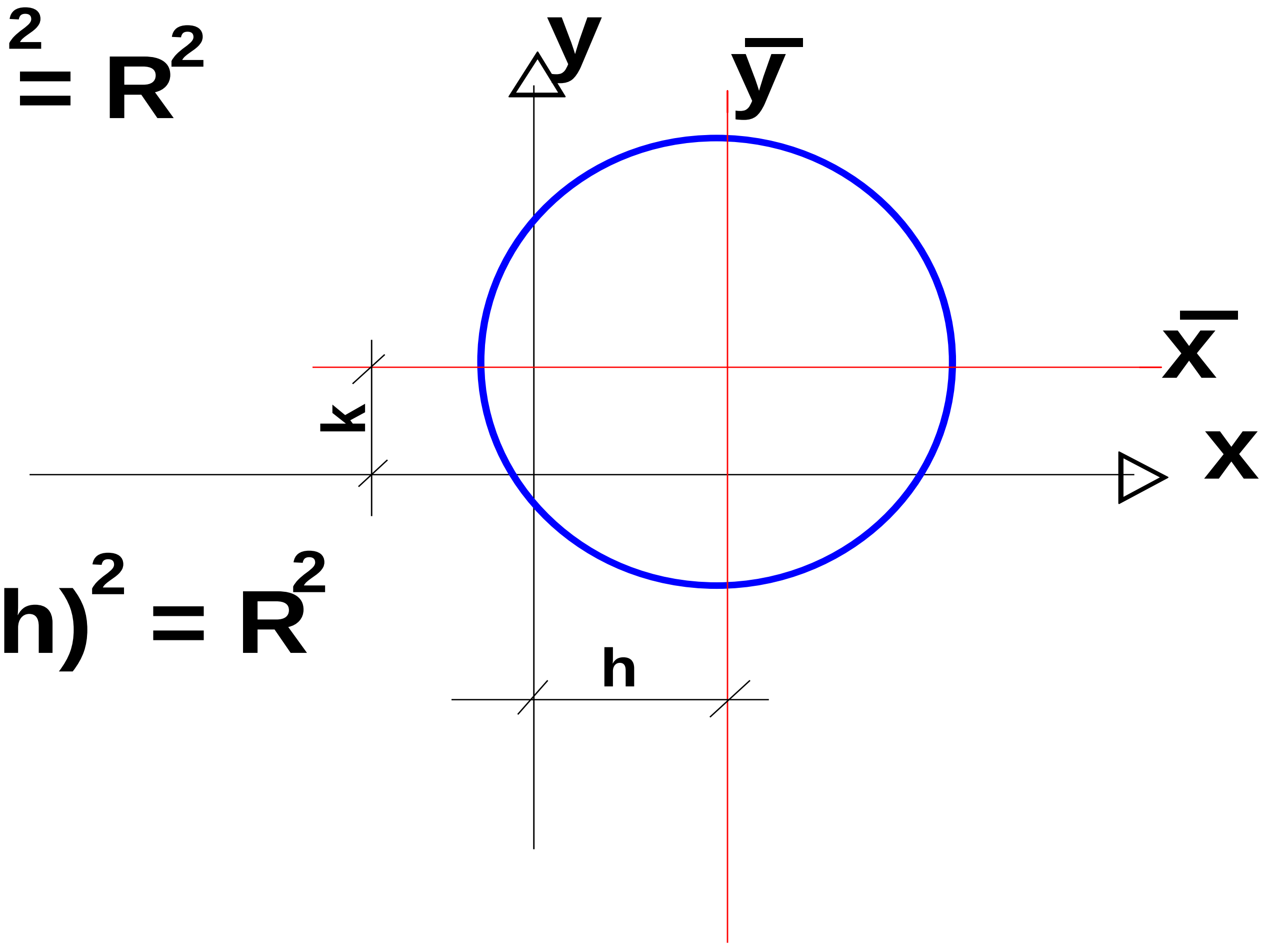
$$x^2 + (y+1)^2 = R^2$$

Circle

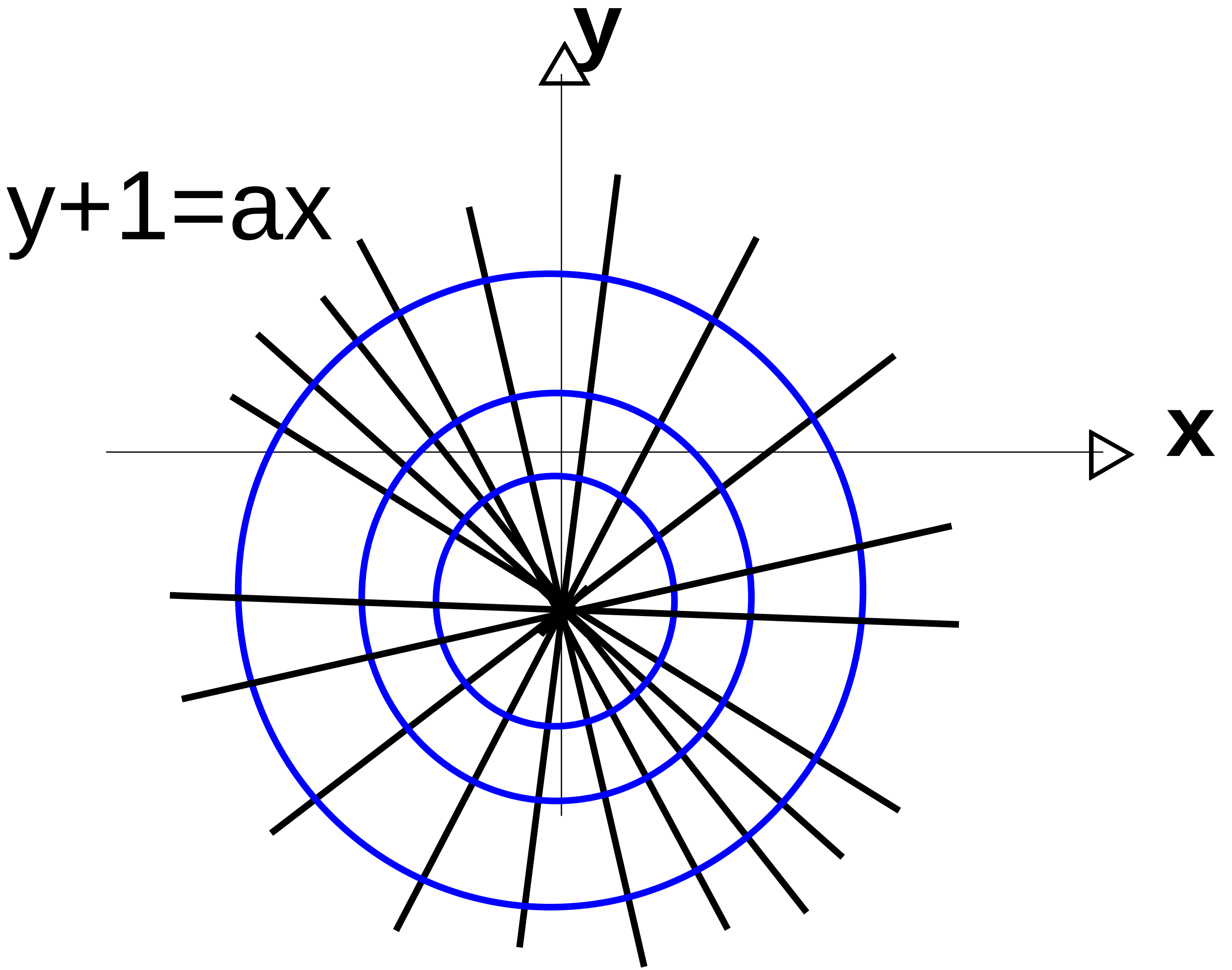
$x^2 + y^2 = R^2$



$$\bar{x}^2 + \bar{y}^2 = R^2$$



$$(x-k)^2 + (y-h)^2 = R^2$$


y+1=ax
y
x

Example: Solve

$$xy(2dx+dy)=2(3xdx+ydy)$$

Solution:

$$(2xy-6x)dx+(xy-2y)dy=0$$

$$2x(y-3)dx+y(x-2)dy=0$$

$$\frac{2x}{x-2}dx+\frac{y}{y-3}dy=0$$

$$(2+\frac{4}{x-2})dx+(1+\frac{3}{y-3})dy=0$$

$$2x+4\ln(x-2)+y+3\ln(y-3)=c$$

$$\ln\left[(x-2)^4(y-3)^3\right]=c-2x-y$$

$$(x-2)^4(y-3)^3=e^{c-2x-y}=e^c e^{-2x-y}$$

$$(x-2)^4(y-3)^3=ke^{-2x-y}$$

Example: Solve

$$(4+y^2)dx+(1+x^2)dy=0$$

Solution:

$$\frac{dx}{1+x^2}+\frac{dy}{4+y^2}=0$$

$$\tan^{-1}x+\frac{1}{2}\tan^{-1}\frac{y}{2}=c_1$$

$$2\tan^{-1}x+\tan^{-1}\frac{y}{2}=2c_1$$

In simpler form:

$$\tan(2\tan^{-1}x+\tan^{-1}\frac{y}{2})=\tan 2c_1$$

$$\frac{\tan 2\tan^{-1}x+\frac{y}{2}}{1-\frac{y}{2}\tan 2\tan^{-1}x}=c_2$$

$$\frac{\frac{2x}{1-x^2}+\frac{y}{2}}{1-\frac{y}{2}\frac{2x}{1-x^2}}=C$$

$$\frac{4x+y(1-x^2)}{2(1-x^2)-2xy}=C$$

Example: Solve

$$dx + xydy = y^2dx + ydy$$

Solution:

Best first step:

$$(1 - y^2)dx = y(1 - x)dy$$

$$\frac{dx}{1 - x} = \frac{y}{1 - y^2}dy$$

$$-\ln(1 - x) = -\frac{1}{2}\ln(1 - y^2) + c$$

$$\ln(1 - x)^2 = \ln(1 - y^2) + C$$

$$\ln\frac{(1 - x)^2}{(1 - y^2)} = C$$

$$\frac{(1-x)^2}{(1-y^2)} = e^C = k^2 \qquad e^C = k^2 \text{ is necessarily +ve}$$

$$(1-x)^2 = k^2(1-y^2) \qquad k \neq 0$$

$$\frac{(1-x)^2}{k^2} = (1-y^2) \qquad k^2 = \lambda$$

$$\frac{(1-x)^2}{\lambda} = (1-y^2) \qquad \lambda \neq 0$$

The general solution defines the *family of conics*:

$$\frac{(x-1)^2}{\lambda} + y^2 = 1$$ (a) general solution

If λ=1 , $$(x-1)^2 + y^2 = 1$$, the solution is a circle

If λ>0 , the solutions are ellipse

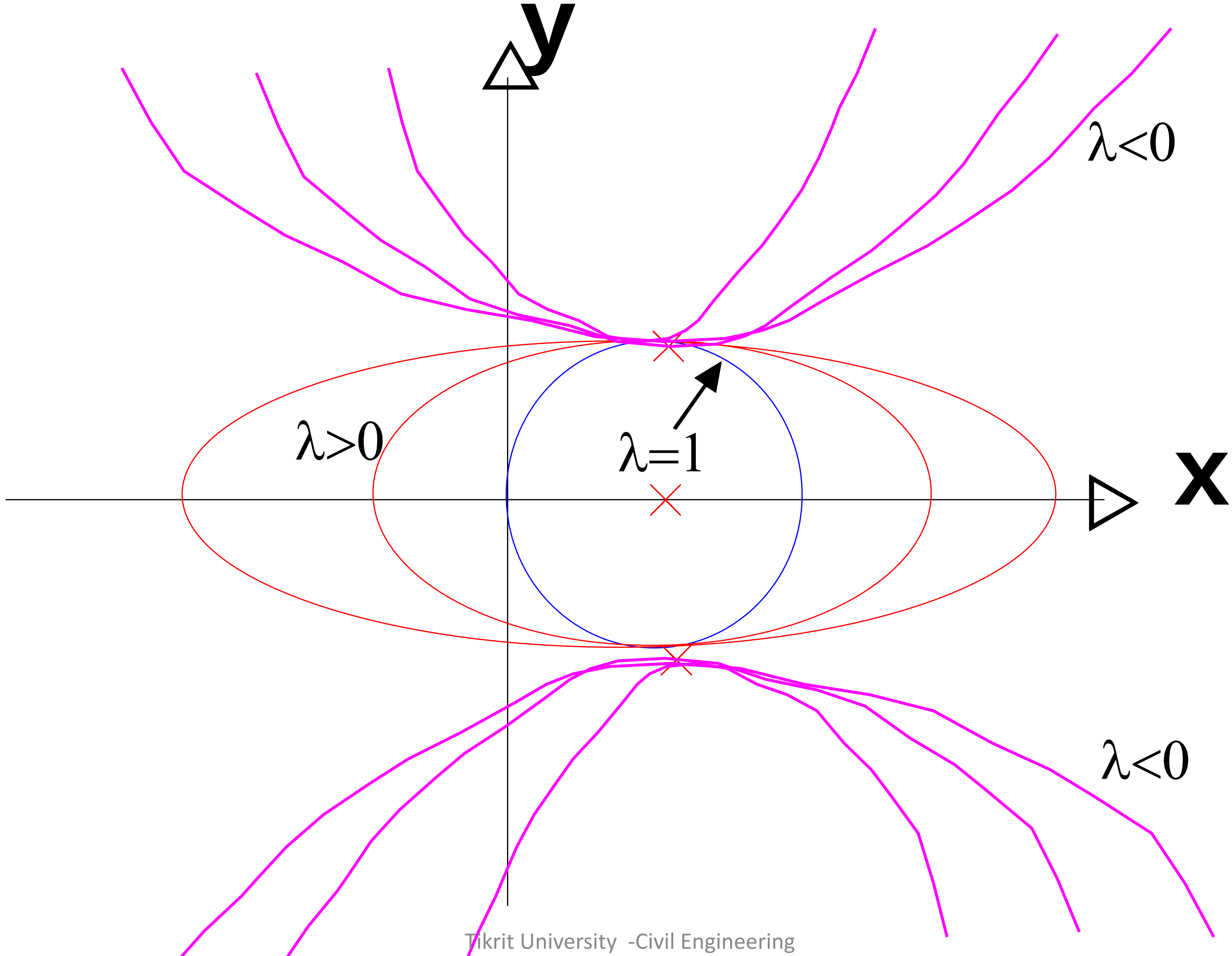
If λ<0 , the solutions are hyperbolas

Particular solution curve which passes through point $(-\frac{7}{5}, \frac{13}{5})$

$$\frac{(-\frac{7}{5}-1)^2}{\lambda}+(\frac{13}{5})^2=1 \Longrightarrow \boxed{\lambda=-1}$$

Hence, particular solution is:

$$\boxed{y^2=1+(x-1)^2} \qquad \text{(b)}$$


y
λ<0
λ>0
λ=1
X
λ<0

The upper branch of any curve of Eq. (a) for $x>0$, can be associated with the upper branch of curve Eq. (b) for $x \le 1$. In the D.E. we divided by $(1-x)$ and $(1-y^2)$, hence $x = 1$, $y = \pm 1$ were implicitly ruled out.

Had we desired the particular solution through $(1, y_o)$, $(x_o, 1)$ or $(x_o, -1)$, we could not have found it from the general solution, even if it is existed.

So we return to the original D.E. and search for the solution by some methods other than separating the variables.

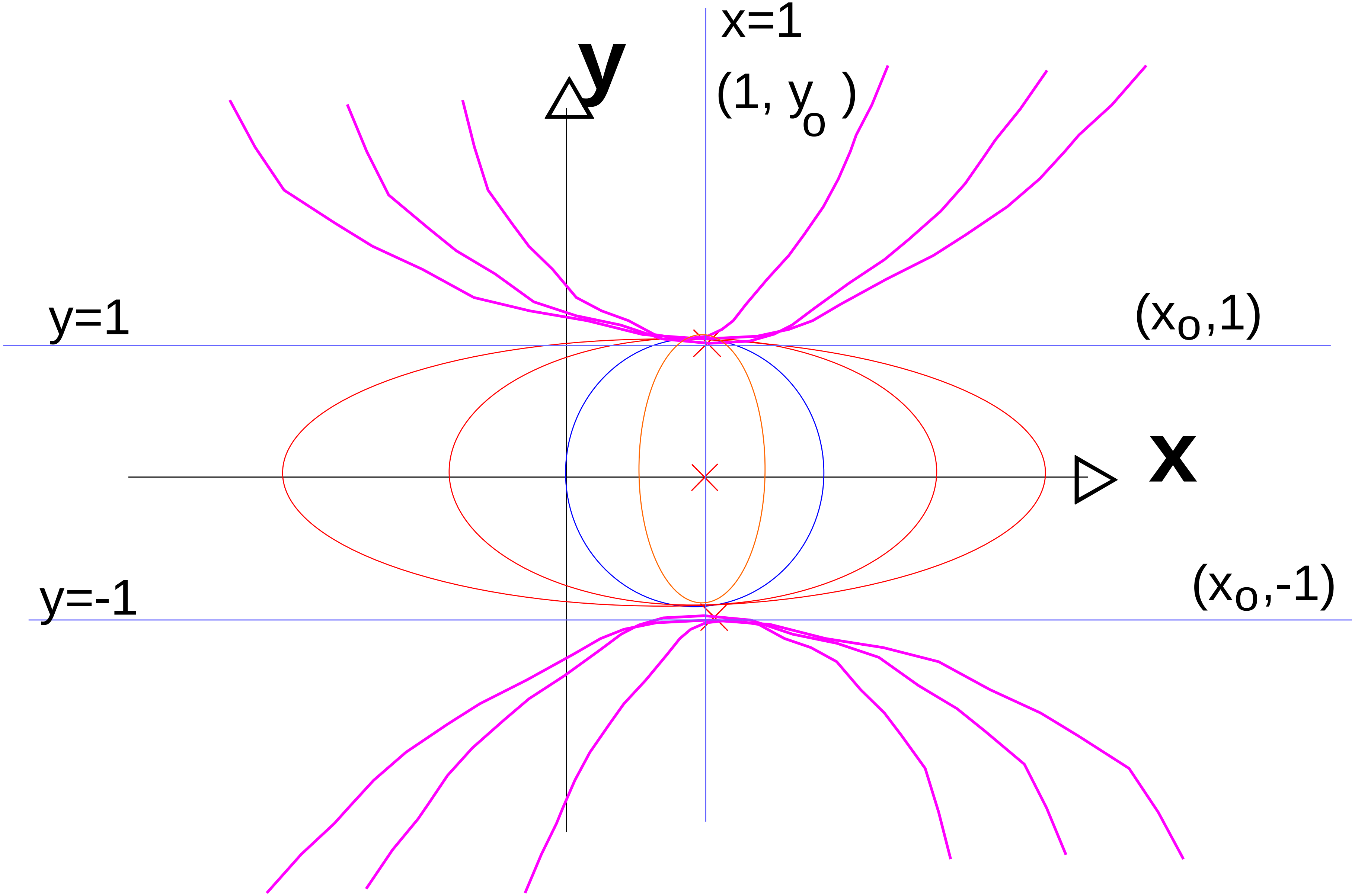
x=1 can be calculated from $(1-x)^2 = \lambda(1-y^2)$

by putting λ=0, So y = 1, y = -1 are singular solutions.

General, $$\frac{(x-1)^2}{\lambda} + y^2 = 1 \qquad \lambda \neq 0$$

Particular, $$y^2 = 1 + (x-1)^2$$

D.E., $$x \neq 1 \quad \frac{dx}{(1-x)} = \frac{ydy}{(1-y^2)} \quad y \neq \pm 1$$


x=1
y
(1, y
o
)
y=1
(x
o
,1)
x
y=-1
(x
o
,-1)

Homogeneous First-Order D.E.

If all terms in M(x,y) and N(x,y) in:

$$M(x,y)dx = N(x,y)dy$$

are all of the same degree in x and y then either of the substitution of $y = ux \quad or \quad x = vy$ will reduce the D.E. to a separable equation.

But, generally if the substitution of

$$x = \lambda x \quad and \quad y = \lambda y$$

will convert $M(x,y)\ \text{into}\ \lambda^n M(x,y)$ and $N(x,y)\ \text{into}\ \lambda^n N(x,y)$, then *M(x,y)* and *N(x,y)* are called homogeneous function of n degree.

The D.E. $M(x,y)dx = N(x,y)dy$ is said homogeneous when *M(x,y)* and *N(x,y)* are homogeneous functions of the same degree.

Example:

$$Is \quad F(x,y) = x(\ln\sqrt{x^2+y^2} - \ln y) + ye^{\frac{x}{y}} \ \text{hom}\,ogeneous\,?$$

Solution:

Substitute $x = \lambda x \quad and \quad y = \lambda y$

$$F(\lambda x, \lambda y) = \lambda x(\ln\sqrt{\lambda^2 x^2 + \lambda^2 y^2} - \ln \lambda y) + \lambda y e^{\frac{\lambda x}{\lambda y}}$$

$$= \lambda x[(\ln\sqrt{x^2+y^2} + \ln\lambda - (\ln\lambda + \ln y)] + \lambda y e^{\frac{x}{y}}$$

$$= \lambda[x(\ln\sqrt{x^2+y^2} - \ln y) + ye^{\frac{x}{y}}]$$

$$= \lambda F(x, y)$$

Hence, homogeneous, degree (1)

OR Subst. either $y = vx \Rightarrow F(x, y) = x F(v)$

or $x = vy \Rightarrow F(x, y) = y F(v)$

Example: Solve $$(x^2+3y^2)dx-2xydy=0$$

Solution:

Is not separable

Is homogeneous of degree (2)

M and N both are of degree (2)

Substitute $y=vx$ $\quad dy=vdx+xdv$

$$(x^2+3v^2x^2)dx-2xvx(vdx+xdv)=0$$

$$x^2(1+3v^2)dx-2x^2v(vdx+xdv)=0$$

$$dx + 3v^2 dx - 2v^2 dx - 2vxdv = 0$$

$$(1 + v^2)dx - 2vxdv = 0$$

$$\frac{1}{x}dx - \frac{2v}{1+v^2}dv = 0$$

$$\ln x - \ln(v^2 + 1) = c$$

$$\ln \frac{x}{v^2 + 1} = c$$

$$\frac{x}{v^2+1}=e^c$$

$$\frac{x}{\left(\frac{y}{x}\right)^2+1}=k$$

$$\frac{x}{\frac{y^2+x^2}{x^2}}=k$$

$$\frac{x^3}{y^2+x^2}=k$$

$$x^3=k(y^2+x^2)$$

Example: Solve $$x^2dt+(x^2-xt+t^2)dx=0$$

Solution: Not separable, but homogeneous.

Substitute $t=vx$ $$dt=vdx+xdv$$

Easier rather than $x=vt$

$$x^2(vdx+xdv)+(x^2-vx^2+v^2x^2)dx=0$$

$$vdx+xdv+dx-vdx+v^2dx=0$$

$$xdv+(1+v^2)dx=0$$

$$\frac{dv}{1+v^2}+\frac{dx}{x}=0$$

$$\tan^{-1}v+\ln x=c$$

$$\ln x=c-\tan^{-1}v$$

$$x=e^{c-\tan^{-1}v}$$

$$x=e^{c}e^{-\tan^{-1}v}$$

$$\boxed{x=k\,e^{-\tan^{-1}\frac{t}{x}}}$$

Example: Solve

$$x\frac{dy}{dx} = y + xe^{-\frac{y}{x}}$$

Solution: Homogeneous.

Substitute $y = vx$ $\quad dy = vdx + xdv$

$$x(vdx + xdv) = (vx + xe^{-\frac{vx}{x}})dx$$

$$vdx + xdv - vdx - e^{-v}dx = 0$$

$$xdv - e^{-v}dx = 0$$

$$\frac{dx}{x} - \frac{dv}{e^{-v}} = 0$$

$$\frac{dx}{x} - e^{v}dv = 0$$

$$\ln x - e^{v} = c$$

$$\ln x - e^{\frac{y}{x}} = c$$

Example: Solve

$$\boxed{(x+y)dx-(3x-y)dy=0}$$

Solution:

Homogeneous

Substitute $y=vx \qquad dy=vdx+xdv$

$$(x+vx)dx-(3x-vx)(vdx+xdv)=0$$

$$(1+v)dx-(3-v)(vdx+xdv)=0$$

$$(1+v)dx-(3vdx+3xdv-v^2dx-vxdv)=0$$

$$(1+v-3v+v^2)dx-(3x-vx)dv=0$$

$$(v^2-2v+1)dx+x(v-3)dv=0$$

$$\frac{dx}{x}+\frac{v-3}{v^2-2v+1}dv=0$$

$$\frac{dx}{x}+\frac{dv}{v-1}-\frac{2}{(v-1)^2}dv=0$$

$$\ln x+\ln(v-1)+\frac{2}{(v-1)}=c$$

$$\ln x(v-1)+\frac{2}{(v-1)}=c$$

$$\ln x(\frac{y}{x}-1)+\frac{2}{(\frac{y}{x}-1)}=c$$

$$\ln(y-x)+\frac{2x}{y-x}=c$$

$$\ln(y-x)=c-\frac{2x}{y-x}$$

$$y-x=e^{c-\frac{2x}{y-x}}$$

$$y-x=ke^{\frac{2x}{x-y}}$$

Exact D.E. of First-Order

If f(x,y) is differentiable, then there is a total differential :

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Conversely if *M(x,y) dx+N(x,y) dy=0* which can be written in the form;

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df = 0$$

With $M(x,y)=\frac{\partial f}{\partial x}$ and $N(x,y)=\frac{\partial f}{\partial y}$

Then $f(x,y)=k$ is a solution.

This D.E. is said to be exact.

But there is a test, in general, for a first order D.E. when it is exact, although sometimes it is not difficult to tell by inspection if the D.E. is exact.

Theorem

If $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous, then

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if and only if $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof:

Assume equation is exact, so there is a function *f*, such that $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ order is immaterial for continuous equations.

Then show that there is a function *f*; $M = \frac{\partial f}{\partial x}$

$$f(x,y) = \int_a^x M(x,y)\,dx + c(y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_a^x M(x,y)\,dx + \bar{c}(y)$$

(but $\int$ and ∂ are interchangeable, since $\frac{\partial M}{\partial x}$ is continuous)

$$\frac{\partial f}{\partial y} = \int_a^x \frac{\partial M}{\partial y} dx + \bar{c}(y)$$

$$\frac{\partial f}{\partial y} = \int_a^x \frac{\partial N}{\partial x} dx + \bar{c}(y)$$

$$Since, \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial y} = N(x,y) - N(a,y) + \bar{c}(y)$$

$$\frac{\partial f}{\partial y} = N(x,y)$$

$$If \quad N(a,y) = \bar{c}(y)$$

$$Hence, \quad c(y) = \int_b^y N(a,y)dy$$

So,

$$f(x,y)=\int_a^x M(x,y)\,dx+\int_b^y N(a,y)dy$$

Is a function

Note:

$$\bar{c}(y)=N-\int_a^x \frac{\partial M}{\partial y}\,dx$$

$$f(x,y)=\int M(x,y)dx+\int [N-\int_a^x \frac{\partial M}{\partial y}\,dx]dy$$

Corollary:

$$If \quad M(x,y)dx + N(x,y)dy = 0 \;\; is\, exact,$$

$$Then, \int_a^x M(x,y)dx + \int_b^y N(a,y)dy = c$$

$Example:$

$Solve;\quad (2x+3y-2)dx+(3x-4y+1)dy=0$

$Solution:$

$(1);$

$$\frac{\partial M}{\partial y}=3 \qquad and \quad \frac{\partial N}{\partial x}=3$$

$\therefore \ exact$

$$\int_a^x (2x+3y-2)\,dx + \int_b^y (3a-4y+1)dy = c$$

$$(x^2+3yx-2x)\Big|_a^x + (3ay-2y^2+y)\Big|_b^y = c$$

$$(x^2+3yx-2x)-(a^2+3ya-2a)$$

$$+(3ay-2y^2+y)-(3ab-2b^2+b)=c$$

$$x^2+3yx-2x-2y^2+y=c+a^2-2a+3ab-2b^2+b$$

$$x^2+3yx-2x-2y^2+y=k$$

$$\therefore\ Solution\quad is\quad x^2-2y^2+3xy-2x+y=k$$

(2);

$$x^2+3yx-2x+\int N(with\,no\;\;x)dy=c$$

$$x^2+3yx-2x+\int(-4y+1)dy=c$$

$$x^2+3yx-2x-2y^2+y=c$$

(3);

$$x^2+3yx-2x+\int[(3x-4y+1)-\int 3dx]dy=c$$

$$x^2+3yx-2x+\int(-4y+1)dy=c$$

$$x^2+3yx-2x-2y^2+y=c$$

Example :

$$Solve;\quad x\sqrt{x^2+y^2}\,dx-\frac{x^2y}{y-\sqrt{x^2+y^2}}\,dy=0$$

Solution :

$$\frac{\partial M}{\partial y}=\frac{xy}{\sqrt{x^2+y^2}}$$

$$and\quad \frac{\partial N}{\partial x}=\frac{xy}{\sqrt{x^2+y^2}}$$

$$\therefore\ exact$$

$$f(x,y)=\int M(x,y)dx+c(y)$$

$$=\int x(x^2+y^2)^{\frac{1}{2}}dx+c(y)$$

$$=\frac{1}{2}(x^2+y^2)^{\frac{3}{2}}\times\frac{2}{3}+c(y)$$

$$=\frac{1}{3}(x^2+y^2)^{\frac{3}{2}}+c(y)$$

$$\bar{c}(y) = N - \int \frac{\partial M}{\partial y} dx$$

$$= \frac{-x^2 y}{y - \sqrt{x^2 + y^2}} - \int \frac{xy}{\sqrt{x^2 + y^2}} dx$$

$$= \frac{-x^2 y}{y - \sqrt{x^2 + y^2}} - \frac{1}{2} y \int \frac{2x}{\sqrt{x^2 + y^2}} dx$$

$$= \frac{-x^2 y}{y - \sqrt{x^2 + y^2}} - \frac{1}{2} y \int 2x (x^2 + y^2)^{-\frac{1}{2}} dx$$

$$\bar{c}(y) = \frac{-x^2 y}{y - \sqrt{x^2 + y^2}} - \frac{1}{2} y (x^2 + y^2)^{\frac{1}{2}} \times \frac{2}{1}$$

$$= \frac{-x^2 y}{y - \sqrt{x^2 + y^2}} - y\sqrt{x^2 + y^2}$$

$$= y[\frac{-x^2}{y - \sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2}]$$

$$= y[\frac{-x^2 - \{y\sqrt{x^2 + y^2} - (x^2 + y^2)\}}{y - \sqrt{x^2 + y^2}}]$$

$$= y[\frac{-x^2 - y\sqrt{x^2 + y^2} + x^2 + y^2}{y - \sqrt{x^2 + y^2}}]$$

$$\bar{c}(y) = y^2\left[\frac{-\sqrt{x^2+y^2}+y}{y-\sqrt{x^2+y^2}}\right]$$

$$\therefore \bar{c}(y) = y^2$$

$$c(y) = \frac{y^3}{3} + k$$

$$\frac{1}{3}(x^2+y^2)^{\frac{3}{2}} + \frac{y^3}{3} = k$$

$$\therefore (x^2+y^2)^{\frac{3}{2}} + y^3 = k$$

D.E. which is not exact can be made exact by multiplying by an integrating factor. For example;

$$2xy^3dx + 3x^2y^2dy = 0$$

is exact, and if it simplified by dividing by *(xy^2),* the equation : $2ydx + 3xdy = 0$

is not exact and can be restored to its original form by multiplying it by factor *(xy^2).* Sometimes the integrating factor can be found by inspection.

Example :

$$(x^2 + y^2 - x)dx - ydy = 0, \ \ show \ \frac{1}{x^2 + y^2}$$

is a factor , and solve.

Solution :

$$(1 - \frac{x}{x^2 + y^2})dx - \frac{y}{x^2 + y^2} dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{2xy}{(x^2 + y^2)^2}$$

$$and \quad \frac{\partial N}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}$$

$\therefore \ \ exact$

$Simpler;$

$$dx - \left(\frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy\right) = 0$$

$$dx - \frac{1}{2}\ \frac{2xdx + 2ydy}{x^2+y^2} = 0$$

$$x - \frac{1}{2}\ln(x^2+y^2) = c$$

$$x - \ln\sqrt{x^2+y^2} = c$$

$$Example:\ ydx+(x^2y^3+x)dy=0,$$
$$find\ \ the\ \ factor\ ,\ \ and\ \ solve.$$

$$Solution$$

$$ydx+xdy+x^2y^3dy=0$$

$$The\ \ factor\ \ is\ \ \frac{1}{(xy)^2}$$

$$\frac{ydx+xdy}{(xy)^2}+\frac{x^2y^3}{(xy)^2}dy=0$$

$$\frac{ydx + xdy}{(xy)^2} + ydy = 0, \ \ or$$

$$\frac{d(xy)}{(xy)^2} + ydy = 0$$

$$-\frac{1}{xy} + \frac{y^2}{2} = c$$

$$Example: xdy - ydx = (4x^2 + y^2)dy,$$

$$find \ the \ integration \ factor, \ and \ solve.$$

Solution

$$The \ factor \ is \ \frac{1}{x^2}$$

$$\frac{xdy - ydx}{x^2} = (4 + \frac{y^2}{x^2})dy$$

$$\frac{xdy - ydx}{x^2(4 + \frac{y^2}{x^2})} = dy$$

$$u=\frac{y}{x},\qquad du=\frac{xdy-ydx}{x^2}$$

$$\frac{du}{(4+u^2)}=dy$$

$$\frac{du}{4[1+(\frac{u}{2})^2]}=dy$$

$$\frac{1}{2}\tan^{-1}\frac{u}{2}=y+c$$

$$\frac{1}{2}\tan^{-1}\frac{y}{2x}=y+c$$

Notes on the Integration Factor :

(1) *If Eq. contains* $xdx + ydy = \left[\frac{1}{2} d(x^2 + y^2) \right]$

$$Try\ (x^2 + y^2)\ \ as\ \ a\ \ multiplier$$

(2) *If Eq. contains* $xdy - ydx = \left[d(\frac{y}{x}) \right]$

$$Try\ \frac{1}{x^2}\ or\ \frac{1}{y^2}\ \ as\ \ a\ \ multiplier$$

(3) *If Eq. contains* $xdy + ydx = \left[d(xy) \right]$

$$Try\ xy\ \ as\ \ a\ \ multiplier$$

Notes :

$$(1)\ d(xy) = xdy + ydx, \qquad \partial f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$(2)\ d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}, \qquad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$(3)\ d(x^2 + y^2) = 2xdx + 2ydy$$

$$(4)\ d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$$

$$(5)\ d\left(\sin^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x\sqrt{x^2 - y^2}}$$

$$(6)\ d(x^m y^n) = mx^{m-1}y^n dx + ny^{n-1}x^m dy$$

$$(7)\ d[\ln(x^2 + y^2)] = \frac{2xdx + 2ydy}{x^2 + y^2}$$

Linear first-order equations

$$F(x)\frac{dy}{dx}+G(x)y=H(x)$$

Divide by $F(x)$ *and rename the coefficien ts;*

$$\frac{dy}{dx}+P(x)y=Q(x)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{d}{dx}[\Phi(x)y]=\Phi(x)\frac{dy}{dx}+\frac{d\Phi(x)}{dx}y\ \ldots\ldots\ldots(2)$$

$$Multiply\ \ Eq.(1)\, by\ \ \Phi(x);$$

$$\Phi(x)\frac{dy}{dx}+\Phi(x)P(x)y=\Phi(x)Q(x)\ .....(3)$$

$$If\ \ \frac{d\Phi(x)}{dx}=\Phi(x)P(x);$$

$$This\ is\ \ a\ simple\ separable\ Eq.$$

$$\frac{d\Phi(x)}{\Phi(x)} = P(x)\,dx$$

$$\ln \Phi(x) = \int P(x)dx$$

$$\Phi(x) = \exp[\int P(x)dx] = e^{\int P(x)dx}$$

$$e^{\int P(x)dx} \ \ is \ \ a \ \ factor$$

$$Multiply \quad Eq.(1)\, by \quad the \quad factor \quad e^{\int P(x)dx};$$

$$e^{\int P(x)dx}[\frac{dy}{dx}+P(x)y]=Q(x)e^{\int P(x)dx}$$

$$\frac{d}{dx}[e^{\int P(x)dx} * y]=Q(x)e^{\int P(x)dx}$$

$$Integrate \quad the \quad above \quad equation;$$

$$ye^{\int P(x)dx}=\int Q(x)e^{\int P(x)dx}dx+c, \quad and \quad finally;$$

$$y=e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx+c\,e^{-\int P(x)dx} \quad ......(4)$$

Steps to solve the linear first-order equations:

1-Compute the integration factor $e^{\int P(x)dx}$

2-Multiply the given equation by this factor.

3-Integrate both sides of the resulting equation.

4-Solve the integrated equation for y.

$$Example : Solve;$$

$$(1+x^2)(dy-dx)=2xydx$$

$$which \quad y=1 \quad when \quad x=0$$

$$Solution :$$

$$Divide \quad the \quad equation \quad by \ \left(1+x^2\right)$$

$$dy-dx=\frac{2xy}{1+x^2}dx$$

$$\frac{dy}{dx}-1=\frac{2xy}{1+x^2}$$

$$\frac{dy}{dx}-\frac{2x}{1+x^2}y=1$$

$$\Phi(x)=e^{-\int\frac{2x}{1+x^2}dx}$$

$$=e^{-\ln(1+x^2)}=e^{\ln\frac{1}{1+x^2}}=\frac{1}{1+x^2}$$

$$\frac{1}{1+x^2}\frac{dy}{dx}-\frac{2x}{(1+x^2)^2}y=\frac{1}{1+x^2}$$

$$By\ \ \text{int}\,egrating;$$

$$\frac{1}{1+x^2}y=\int\frac{1}{1+x^2}dx+c$$

$$\frac{1}{1+x^2}y=\tan^{-1}x+c$$

$$y=(1+x^2)\tan^{-1}x+c(1+x^2)$$

$$y=1 \qquad x=0$$

$$1=0+c(1+0)$$

$$\therefore c=1$$

$$\therefore \quad y=(1+x^2)\tan^{-1}x+(1+x^2)$$

$$Example: Solve;$$

$$ydx + (xy + 2x - 2y)dy = 0$$

$$Solution:$$

$$ydx + xydy + 2xdy - 2ydy = 0$$

$$y\frac{dx}{dy} + xy + 2x = 2y$$

$$y\frac{dx}{dy} + (y+2)x = 2y$$

$$\frac{dx}{dy} + \frac{y+2}{y}x = 2$$

$$\Phi(y) = e^{\int \frac{y+2}{y} dy} = e^{y+2\ln y} = e^{y} y^{2}$$

$$y^{2} e^{y} x = \int 2y^{2} e^{y} dy$$

$$= 2(y^{2} e^{y} - 2ye^{y} + 2e^{y}) + c$$

$$xy^{2} = 2y^{2} - 4y + 4 + ce^{-y}$$

The Bernoulli's Equation:

D.E. of the form

$$\frac{dy}{dx} + Py = Q\,y^{n}$$

is said to be a *Bernoulli's equation*. (*P* and *Q* are functions of *x* (or constants) and do not contain *y*)

$$\frac{dy}{dx} + Py = Q\, y^n$$

By dividing both sides by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + Py^{1-n} = Q$$

$$v = y^{1-n} \qquad \frac{dv}{dx} = (1-n)\frac{1}{y^n}\frac{dy}{dx}$$

$$\frac{1}{1-n}\frac{dv}{dx}+Pv=Q$$

$$\frac{dv}{dx}+(1-n)Pv=Q(1-n)$$

$$If\ \ n=0 \quad \Rightarrow linear \quad first\ order$$

$$If\ \ n=1 \quad \Rightarrow separable \ \ first\ order$$

$$Bernoulli's\ Eq. \quad n\neq 0\, and\, 1$$

$Example : Solve; \quad \frac{dy}{dx}+\frac{y}{x}=y^3$

$Solution :$

$$\frac{dy}{dx}+\frac{y}{x}=y^3 \qquad is \;\; Ber. \;\; Eq.$$

$$\frac{1}{y^3}\frac{dy}{dx}+\frac{1}{xy^2}=1$$

$$Put \quad \frac{1}{y^2}=v \Rightarrow -\frac{2}{y^3}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\therefore \frac{1}{y^3}\frac{dy}{dx}=-\frac{1}{2}\frac{dv}{dx}$$

$$-\frac{1}{2}\frac{dv}{dx}+\frac{v}{x}=1$$

$$\frac{dv}{dx}-\frac{2}{x}v=-2 \quad linear \quad in \quad v$$

$$\Phi(x)=e^{\int P(x)dx}=e^{-2\int\frac{dx}{x}}=e^{-2\ln x}$$

$$=e^{\ln x^{-2}}=\frac{1}{x^2}$$

$$v\frac{1}{x^2}=\int -2e^{\int P(x)dx}dx+c$$

$$v\frac{1}{x^2}=\int \frac{-2}{x^2}dx+c$$

$$v\frac{1}{x^2}=\frac{2}{x}+c$$

$$\frac{1}{y^2x^2}=\frac{2}{x}+c$$

$$y^{-2} = 2x + cx^2$$

$$1 = xy^2(2 + cx)$$

$$\therefore x\, y^2(2 + cx) = 1$$

$$Example : Solve; \quad \frac{dy}{dx} - \frac{y}{x+1} = e^x y^3$$

$$Solution :$$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^x y^3 \qquad is \;\; Ber. \;\; Eq.$$

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{1}{x+1}\frac{1}{y^2} = e^x$$

$$Put \quad \frac{1}{y^2} = v \Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

$$-\frac{1}{2}\frac{dv}{dx}-\frac{v}{x+1}=e^{x}$$

$$\frac{dv}{dx}+\frac{2}{x+1}v=-2e^{x} \quad linear \quad in \quad v$$

$$\Phi(x)=e^{\int P(x)dx}=e^{2\int\frac{dx}{x+1}}=e^{2\ln(x+1)}$$

$$=e^{\ln(x+1)^{2}}=(x+1)^{2}$$

$$v(x+1)^2 = \int -2e^x (x+1)^2 dx + c$$

$$= -2\int (x+1)^2 e^x dx + c$$

$$v(x+1)^2 = -2[(x+1)^2 e^x - 2(x+1)e^x + 2e^x] + c$$

$$v(x+1)^2 = -2(x+1)^2 e^x + 4(x+1)e^x - 4e^x + c$$

$$v = -2e^x + \frac{4e^x}{x+1} - \frac{4e^x}{(x+1)^2} + \frac{c}{(x+1)^2}$$

$$\therefore \frac{1}{y^2} = -2e^x + \frac{4e^x}{x+1} - \frac{4e^x}{(x+1)^2} + \frac{c}{(x+1)^2}$$

$$Example: Solve; \quad \frac{dy}{dx} + xy = y^2 e^{\frac{x^2}{2}} \sin x$$

Solution :

$$\frac{dy}{dx} + xy = y^2 e^{\frac{x^2}{2}} \sin x \qquad is \quad Ber. \quad Eq.$$

$$\frac{1}{y^2}\frac{dy}{dx} + x\frac{1}{y} = e^{\frac{x^2}{2}} \sin x$$

$$Put \quad \frac{1}{y} = v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{1}{y^2}\frac{dy}{dx} = -\frac{dv}{dx}$$

$$-\frac{dv}{dx}+xv=e^{\frac{x^2}{2}}\sin x$$

$$\frac{dv}{dx}-xv=-e^{\frac{x^2}{2}}\sin x \quad linear\ \ D.E.\ with\ \ v$$

$$\Phi(x)=e^{\int -xdx}=e^{-\frac{x^2}{2}}$$

$$e^{-\frac{x^2}{2}}v=\int e^{-\frac{x^2}{2}}(-e^{\frac{x^2}{2}}\sin x)dx+c$$

$$e^{-\frac{x^2}{2}}v=-\int \sin x dx+c$$

$$e^{-\frac{x^2}{2}} v = \cos x + c$$

$$e^{-\frac{x^2}{2}} \frac{1}{y} = \cos x + c$$

$$\therefore \quad \frac{1}{y} = e^{\frac{x^2}{2}} (\cos x + c)$$

$$When\ \ t_o = 0\ ,\ \ Q = 100\ lb$$

$$100 = 200 + c$$

$$\therefore c = -100$$

$$Q = 200 - 100\, e^{-\frac{t}{20}}$$

$$When\ Q = 150\, lb;$$

$$150 = 200 - 100\, e^{-\frac{t}{20}}$$

$$\therefore t = 13.86\ \text{min}$$

Orthogonal Trajectories

$$Slope = \left(\frac{dy}{dx}\right)_{original} \times \left(\frac{dy}{dx}\right)_{orthogonal} = -1$$

$$\therefore \left(\frac{dy}{dx}\right)_{orthogonal} = \frac{-1}{\left(\frac{dy}{dx}\right)_{original}}$$

Example : Given;

$$y^2 = 4cx$$

Find orthogonal trajectories.

Solution:

$$y^2 = 4cx \Rightarrow c = \frac{y^2}{4x}$$

$$2yy' = 4c = 4\frac{y^2}{4x} = \frac{y^2}{x}$$

$$\therefore\ y' = \frac{y}{2x}$$

Slope of orthogonal:

$$y' = \frac{-2x}{y}$$

$$y' = \frac{-2x}{y}$$

$$\frac{dy}{dx} = -\frac{2x}{y}$$

$$ydy = -2xdx$$

$$\frac{y^2}{2} = -x^2 + c$$

$$\therefore \frac{y^2}{2} + x^2 = c$$

Example: A hemispherical tank of radius; *R,* is initially filled with water. At the bottom of the tank there is a hole or radius; *r,* through which water drains under the influence of gravity. Find the depth of the water at any time; t, and determine how long it will be taken the tank to drain completely.

$Solution:$

$Decrease\ in\ the\ volume\ of\ water;$

$$dV = \pi\ x^2\, dy$$

$Volume\ of\ water\ with\ \text{int}erval\ dt,\ stream$

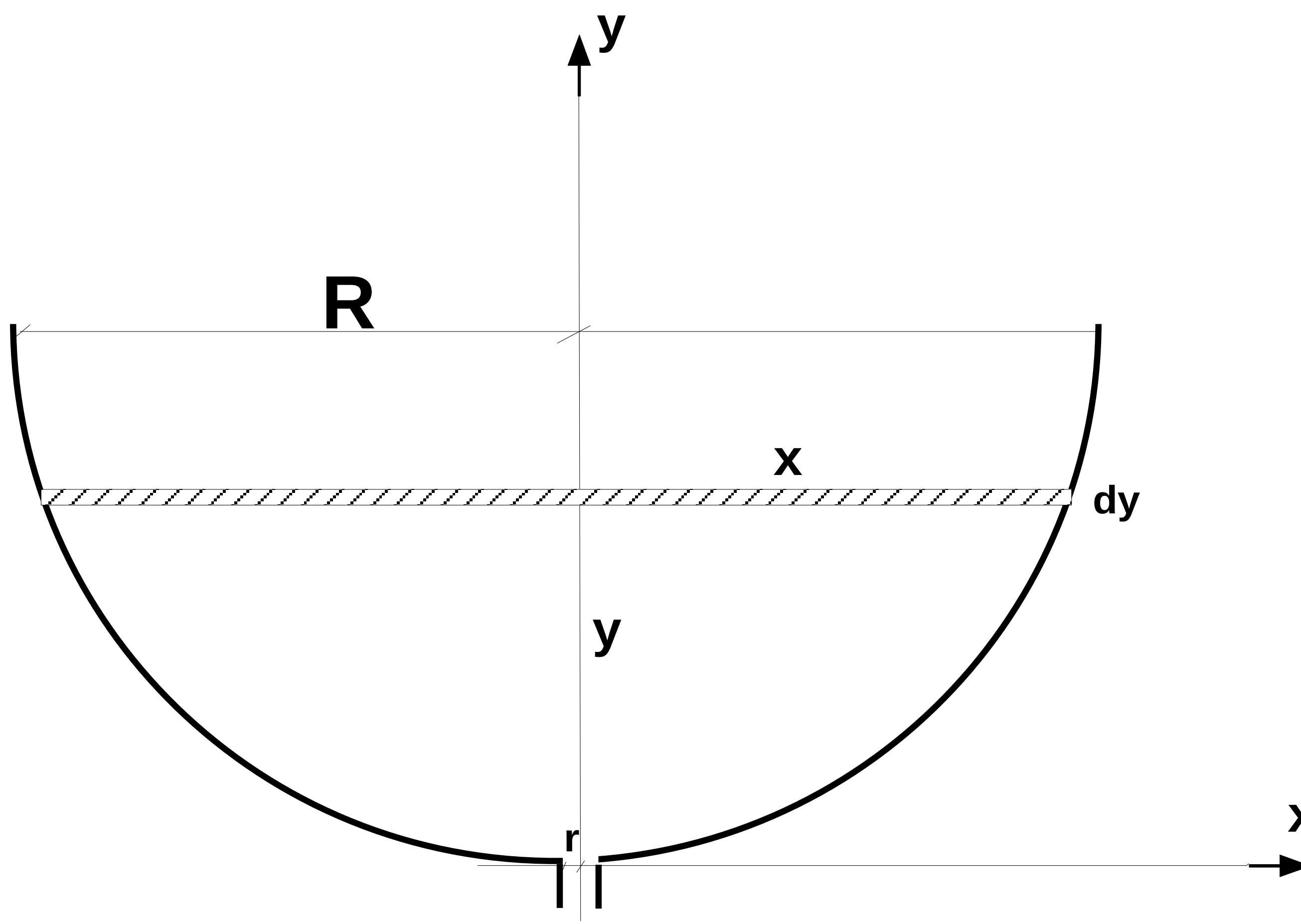
$of\ water\ vdt\ and\ A = \pi r^2$

$$dV = vAdt$$

$$v = \sqrt{2gh} \quad from\ orifice\ (Torricelli\ law)$$

$$g = accelaration\ gravity$$

$$h = ins\tan\tan eous\ height\ (head)$$


y
R
x
dy
y
r
x

$$dV = \pi \; x^2 \, dy \qquad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

$$dV = \pi \, r^2 \sqrt{2gy} \; dt \; \dots\dots\dots\dots\dots\dots\dots(2)$$

$$Equating \quad Eqs. \; (1) \, \& \, (2)$$

$$\pi \; x^2 \, dy = - \; \pi \, r^2 \sqrt{2gy} \; dt$$

$$(2yR - y^2) dy = -r^2 \sqrt{2gy} \; dt$$

$$(2y^{\frac{1}{2}} R - y^{\frac{3}{2}}) dy = -r^2 \sqrt{2g} \; dt$$

$$\frac{4}{3}y^{\frac{3}{2}}R-\frac{2}{5}y^{\frac{5}{2}}=-r^{2}\sqrt{2g}\ t+c$$

$$When\ t=0,\ y=R$$

$$\frac{4}{3}R^{\frac{5}{2}}-\frac{2}{5}R^{\frac{5}{2}}=0+c$$

$$c=\frac{14}{15}R^{\frac{5}{2}}$$

$$\frac{4}{3}y^{\frac{3}{2}}R-\frac{2}{5}y^{\frac{5}{2}}=-r^{2}\sqrt{2g}\;t+\frac{14}{15}R^{\frac{5}{2}}$$

$$No\ \ water;\ \ y=0,\ \ t=??$$

$$0=-r^{2}\sqrt{2g}\;t+\frac{14}{15}R^{\frac{5}{2}}$$

$$t=\frac{14}{15}\frac{R^{\frac{5}{2}}}{r^{2}\sqrt{2g}}$$

Example: The rate at which a solid substance dissolves varies directly as the amount of undissolved solid presented in the solvent and as the difference between the instantaneous concentration and the saturation concentration of the substance. *Twenty pounds* of solute is dumped into a tank containing *120 lb* of solvent, and at the end of *12 min*. the concentration is observed to be *1 part in 30*. Find the amount of solute in solution at any time; t, if the saturation concentration is 1 part of solute to 3 parts of solvent.

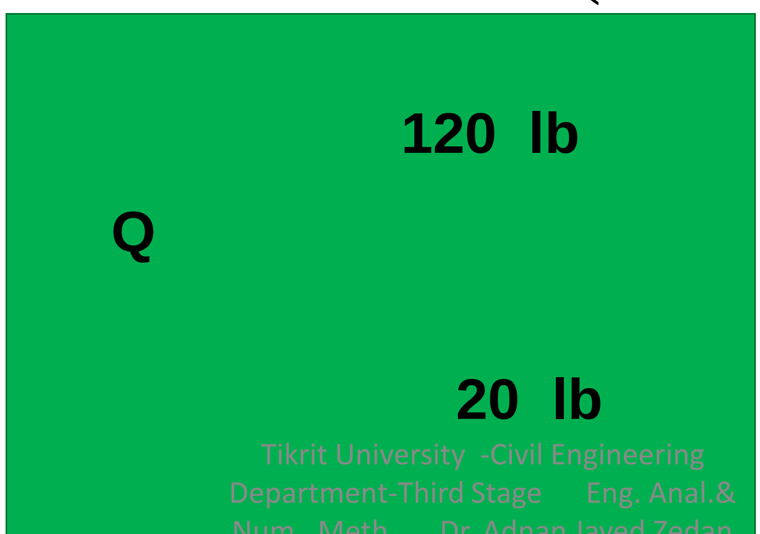
Solution :

If Q *is the amount of the material in solution at time;* t,

$\therefore (20-Q)$ *is the amount of undissolved material at that time*

$\dfrac{Q}{120}$ *is the corresponding concentration*

$$\frac{dQ}{dt} \propto undissolved\ \ material \times (saturation - concentration)$$



$$\frac{dQ}{dt} = k\,(20 - Q)\,(\frac{1}{3} - \frac{Q}{120})$$

$$\frac{dQ}{dt} = \frac{k}{120}(20 - Q)\,(40 - Q)$$

$$\frac{dQ}{(20 - Q)(40 - Q)} = \frac{k}{120}dt$$

$$\int [\frac{1}{20(20 - Q)} - \frac{1}{20(40 - Q)}]dQ = \int \frac{k}{120}dt + c$$

$$\int [\frac{1}{20(20-Q)} - \frac{1}{20(40-Q)}]dQ = \int \frac{k}{120}dt + c$$

$$\frac{1}{20}\int [\frac{1}{(20-Q)} - \frac{1}{(40-Q)}]dQ = \frac{1}{20}\int \frac{k}{6}dt + c$$

$$-\ln(20-Q) + \ln(40-Q) = \frac{k}{6}t + c$$

$$\ln\frac{40-Q}{20-Q} = \frac{k}{6}t + c$$

$$\ln\frac{40-Q}{20-Q}=\frac{k}{6}t+c$$

$$When \quad t=0 \ , \ Q=0$$

$$\ln\frac{40-0}{20-0}=\frac{k}{6}(0)+c \quad \Rightarrow \ c=\ln 2$$

$$\therefore \ln\frac{40-Q}{20-Q}=\frac{k}{6}t+\ln 2, \quad or$$

$$\ln\frac{40-Q}{2(20-Q)}=\frac{k}{6}t$$

$$\ln \frac{40-Q}{2(20-Q)} = \frac{k}{6} t$$

$$After\ \ 12\,\text{min}.\ \ concet. = \frac{1}{30} = \frac{Q}{120}\ \Rightarrow\ Q = 4$$

$$\ln \frac{40-4}{2(20-4)} = \frac{k}{6}(12) \Rightarrow k = 0.05889$$

$$\therefore \ln \frac{40-Q}{2(20-Q)} = 0.0098\ t$$

$$\frac{40-Q}{40-2Q} = e^{0.0098t}$$

$$\therefore Q = \frac{40(1-e^{-0.0098t})}{2-e^{-0.0098t}}$$