The Laplace Transformation

Let $f(t)$ be given and defined for $t \geq 0$,

Multiply $f(t)$ by e^{-st} and integrate with respect to t from 0 to ∞, is a function of S , Say

$f(s)$;

$$f(s) = \int_0^\infty e^{-st} f(t)dt \qquad \textit{Laplace....} \qquad (1)$$

Transform of $f(t) = \mathcal{L}(f)$

The operation is called the Laplace transformation

$$\mathrm{f}(t) = \mathcal{L}^{-1}(F) = inverse\ transform \quad or\ inverse$$

$\underline{Theorem}:$ Let $f(t)$ be a piecewise continuous on every finite interval in the range $t \geq 0$, and satisfies :

$$|f(t)| \leq Me^{\gamma t} \quad for \ all \quad t \geq 0$$

and for some constants γ and M.

Then the Laplace form 1, exists for $s > \gamma$.

Proof :

$$|\mathcal{L}(f)| = \left| \int_0^\infty e^{-st} f(t)\,dt \right|$$

$$\le \int_0^\infty e^{-st} |f(t)|\,dt$$

$$\le \int_0^\infty e^{-st} M e^{\gamma t}\,dt$$

$$= M \int_0^\infty e^{-(s-\gamma)t}\,dt = \frac{M}{s-\gamma}$$

Examples:

$$|\mathcal{L}(1)| = \int_0^\infty 1e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad If \ \ s>0$$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}\,e^{-st}dt = \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty = \frac{1}{s-a} \qquad If \ \ s-a>0$$

$$\mathcal{L}(e^{-at}) = \int_0^\infty e^{-at}\,e^{-st}dt = \frac{e^{-(s+a)t}}{-(s+a)}\Big|_0^\infty = \frac{1}{s+a} \qquad If \ \ s+a>0$$

$$\mathcal{L}(\cosh\ at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2} \qquad If \ \ s>a$$

$$\mathcal{L}(\sin\ wt) = \frac{w}{s^2 + w^2}$$

$$\mathcal{L}(\cos\ wt) = \frac{s}{s^2 + w^2}$$

$$\mathcal{L}(t^2) = \frac{2!}{s^3}$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \qquad (n = 1,2,3,\ldots\ldots)$$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$$\mathcal{L}(\sinh\, a\,t) = \frac{a}{s^2 - a^2}$$

$$\mathcal{L}\frac{1}{w^2}(1 - \cos wt) = \frac{1}{s(s^2 + w^2)}$$

$$Set \quad a = i\,w$$

$$\mathcal{L}(e^{at}) = \mathcal{L}\left(e^{iwt}\right) = \frac{1}{s - iw} = \frac{s + iw}{s^2 + w^2}$$

$$= \frac{s}{s^2+w^2} + i\,\frac{w}{s^2+w^2}$$

$$But \quad e^{iwt} = \cos wt + i \sin wt$$

$$\mathcal{L}e^{iwt} = \mathcal{L}\cos wt + i\,\mathcal{L}\sin wt$$

$$= \frac{s}{s^2 + w^2} + i\,\frac{w}{s^2 + w^2}$$

$$\mathcal{L}\left(f^{(n)}\right) = s^n\mathcal{L}(f) - s^{n-1}f_o - s^{n-2}f'_o - s^{n-3}f''_o \dots\dots\dots\dots f_o^{(n-1)}$$

$$(f''') = s^3\mathcal{L}(f) - s^2 f_o - sf'_o - f''_o$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f_o$$

$$Example: \quad \mathcal{L}(y) = \frac{s+1}{s^2+s-6}, \qquad find \;\; y(t)$$

<u>*Solution:*</u>

$$\frac{s+1}{s^2+s-6} = \frac{s+1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$= \frac{A(s+3)+B(s-2)}{(s-2)(s+3)}$$

$s+1 = A(s+3) + B(s-2),$ $\qquad$ $setting \;\; s = 2, and \;\; then \;\; s = -3$

$$find \qquad A = \frac{3}{5} \qquad and \qquad B = \frac{2}{5}$$

$$L\{y(t)\} = \frac{1}{5}\left(\frac{3}{s-2} + \frac{2}{s+3}\right)$$

$$\therefore\ y(t) = \frac{1}{5}(3e^{2t} + 2e^{-3t})$$

$$Example: \quad f(t) = sin^2 t \,, \qquad find \ \mathcal{L}(f)$$

Solution:

$$f'(t) = 2\sin t \ \cos t = \sin 2t \quad , \qquad f(0) = 0$$

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2+4} \,,$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(o)$$

$$\frac{2}{s^2+4} = s\mathcal{L}(f) - 0$$

$$\therefore \mathcal{L}(f) = \frac{2}{s(s^2+4)}$$

$$Example: find\ \ the\ \ particular\ \ solution\ \ of$$

$$y'' - 3y' + 2y = 12e^{-2t} \qquad y(0) = 2\ \&\ y'(0) = 6$$

Solution:

$$\mathcal{L}y'' - 3\mathcal{L}y' + 2\mathcal{L}y = 12\mathcal{L}e^{-2t}$$

$$s^2\mathcal{L}y - sf_o - f'_o - 3(s\mathcal{L}y - f_o) + 2\mathcal{L}y = 12\mathcal{L}e^{-2t}$$

$$s^2\mathcal{L}y - s(2) - 6 - 3(s\mathcal{L}y - 2) + 2\mathcal{L}y = 12\frac{1}{s+2}$$

$$(s^2 - 3s + 2)\mathcal{L}y = 2s + \frac{12}{s+2} = \frac{2s^2 + 4s + 12}{s+2}$$

$$(s^2 - 3s + 2)\mathcal{L}y = \frac{2s^2 + 4s + 12}{s + 2}$$

$$\therefore \mathcal{L}y = \frac{2s^2 + 4s + 12}{(s^2 - 3s + 2)(s + 2)} = \frac{2s^2 + 4s + 12}{(s - 2)(s - 1)(s + 2)}$$

$$= \frac{A}{(s - 2)} + \frac{B}{(s - 1)} + \frac{C}{(s + 2)}$$

$$= \frac{7}{(s - 2)} - \frac{6}{(s - 1)} + \frac{1}{(s + 2)}$$

$$\therefore y(t) = 7e^{2t} - 6e^{t} + e^{-2t}$$

$Example$: $Solve$;

$$y'' + 4y' + 3y = 0 \qquad y(0) = 3 \ \& \ y'(0) = 1$$

$\underline{Solution}$:

$$\mathcal{L}y'' + 4\mathcal{L}y' + 3\mathcal{L}y = 0$$

$$s^2\mathcal{L}y - sf_o - f'_o + 4(s\mathcal{L}y - f_o) + 3\mathcal{L}y = 0$$

$$s^2\mathcal{L}y - 3s - 1 + 4(s\mathcal{L}y - 3) + 3\mathcal{L}y = 0$$

$$(s^2 + 4s + 3)\mathcal{L}y = 3s + 1 + 12$$

$$\therefore \mathcal{L}y = \frac{3s+13}{(s^2+4s+3)} = \frac{3s+13}{(s+3)(s+1)}$$

$$= \frac{A}{s+3} + \frac{B}{s+1}$$

$$= \frac{-2}{s+3} + \frac{5}{s+1}$$

$$\therefore y(t) = -2e^{-3t} + 5e^{-t}$$

$\underline{Note:}$

$$If \quad \mathcal{L}\{f(t)\} = F(s)$$

$$Then; \quad \mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$Example$: $Solve$;

$$y'' + 2y' + 5y = 0 \qquad y(0) = 2 \;\&\; y'(0) = -4$$

$Solution$:

$$\mathcal{L}y'' + 2\mathcal{L}y' + 5\mathcal{L}y = 0$$

$$s^2\mathcal{L}y - sf_o - f'_o + 2(s\mathcal{L}y - f_o) + 5\mathcal{L}y = 0$$

$$s^2Y - 2s + 4 + 2(sY - 2) + 5Y = 0 \qquad \boxed{Y = \mathcal{L}(y)}$$

$$(s^2 + 2s + 5)Y = 2s$$

$$\therefore Y(s) = \mathcal{L}(y) = \frac{2s}{(s^2 + 2s + 5)} = \frac{2s}{(s+1)^2 + 2^2}$$

$$= 2\frac{s+1}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 2^2}$$

$$But; \ \mathcal{L}^{-1}\left(\frac{s}{s^2 + 2^2}\right) = \cos 2t \quad \& \quad \mathcal{L}^{-1}\left(\frac{2}{s^2 + 2^2}\right) = sin\, 2t$$

$$\therefore y(t) = \mathcal{L}^{-1}(y) = e^{-t}(2\cos 2t - \sin 2t)$$

Inverse Laplace Transforms:

$$\because \mathcal{L}\{f(t)\} = F(s)$$

$$\therefore f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Example:

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{2t}$$

$F(s)$	$\mathcal{L}^{-1}F(s) = f(t)$
$\dfrac{1}{s}$	1
$\dfrac{1}{s^2}$	t
$\dfrac{1}{s^{n+1}}$	$\dfrac{t^n}{n!}$
$\dfrac{1}{s-a}$	e^{at}

$F(s)$	$\mathcal{L}^{-1}F(s) = f(t)$
$\dfrac{1}{s^2 + a^2}$	$\dfrac{\sin at}{a}$
$\dfrac{s}{s^2 + a^2}$	$\cos at$
$\dfrac{1}{s^2 - a^2}$	$\dfrac{\sinh at}{a}$
$\dfrac{s}{s^2 - a^2}$	$\cosh at$

Examples:

$$(1)\quad \mathcal{L}^{-1}\left(\frac{5}{s+3}\right) = 5e^{-3t}$$

$$(2)\quad F(s) = \frac{s+1}{s^2+2} \quad find \quad f(t)$$

$$F(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore f(t) = \cos t + \sin t$$

$$(3) \quad F(s) = \frac{1}{(s+25)^2} \qquad find \quad f(t)$$

$$Note: \ \mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

$$F(s) = \frac{1}{s^2} \quad , \qquad a = -25$$

$$f(t) = t$$

$$\therefore The \quad result\ is \quad te^{-25t}$$

$$(4)\quad F(s)=\frac{s+2}{(s+2)^2+1} \qquad find \quad f(t)$$

$$f(t)=e^{-2t}\cos t$$

$$(5)\quad F(s)=\frac{s}{(s-1)^2-4} \qquad find \quad f(t)$$

$$\frac{s-1+1}{(s-1)^2-4}=\frac{s-1}{(s-1)^2-4}+\frac{1}{(s-1)^2-4}$$

$$\therefore f(t)= e^{t}\cosh 2t + \frac{1}{2}e^{t}\sinh 2t$$