Partial Differential Equations:

Basic Concepts:

An equation involving one or more partial derivatives of an (unknown) function of two or more independent variables is called a partial differential equation. The order of the highest derivative is called the order of the equation.

Just as in the case of an ordinary differential equation, we say that a partial differential equation is linear if it is of the first degree in the dependent variable (the unknown function) and its partial derivatives. If each term of such term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be homogeneous; otherwise it is said to be nonhomogeneous.

Example :

Important linear partial differential equation of the second order :

(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ One-dimensional wave Eq.

(2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ One-dimensional heat Eq.

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Two-dimensional Laplace Eq.

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ Two-dimensional Poisson Eq.

(5) $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ Two-dimensional wave Eq.

(6) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ Three-dimensional Laplace Eq.

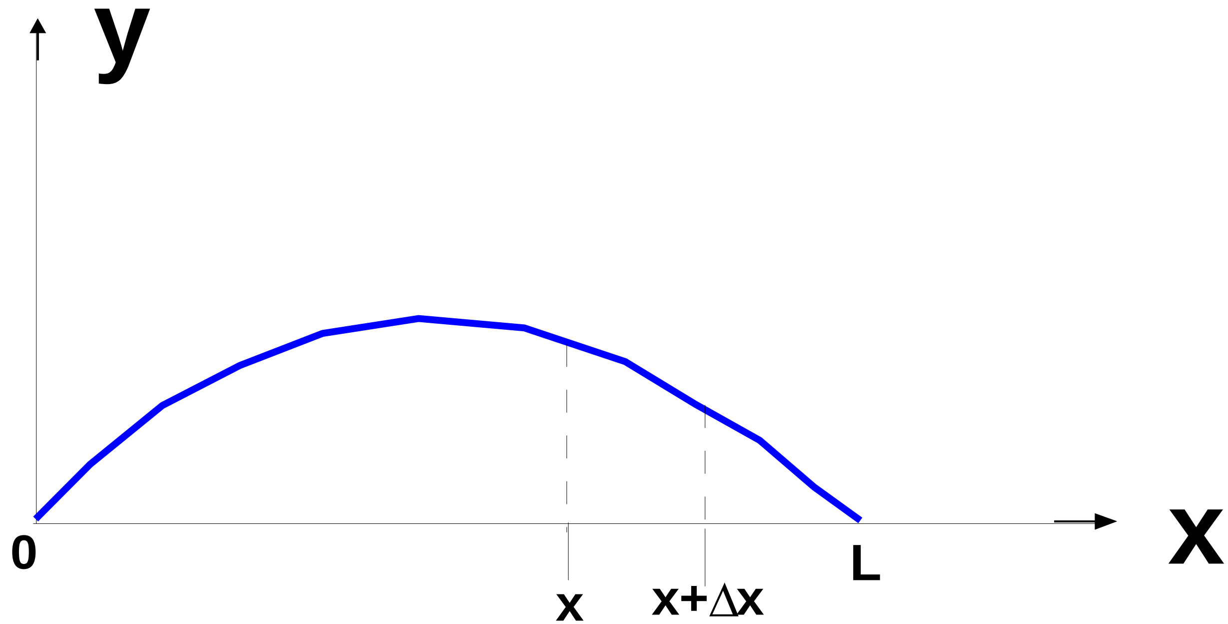
Here (c) is a constant; (t) is a time; x, y, z are the Cartesian coordinates and the dimension is the number of these coordinates in the equation. Equation (4) [with $f(x,y)\neq 0$] is nonhomogeneous, while the other equations are homogeneous.

MODELING:

(1)Vibration of elastic string (wave equation):

Assumptions;

- Vibration takes place in x-y plane.
- No elongation.
- The string can transmit force only in the direction of its length.
- Constant tension force.


y
0
x
x+Δx
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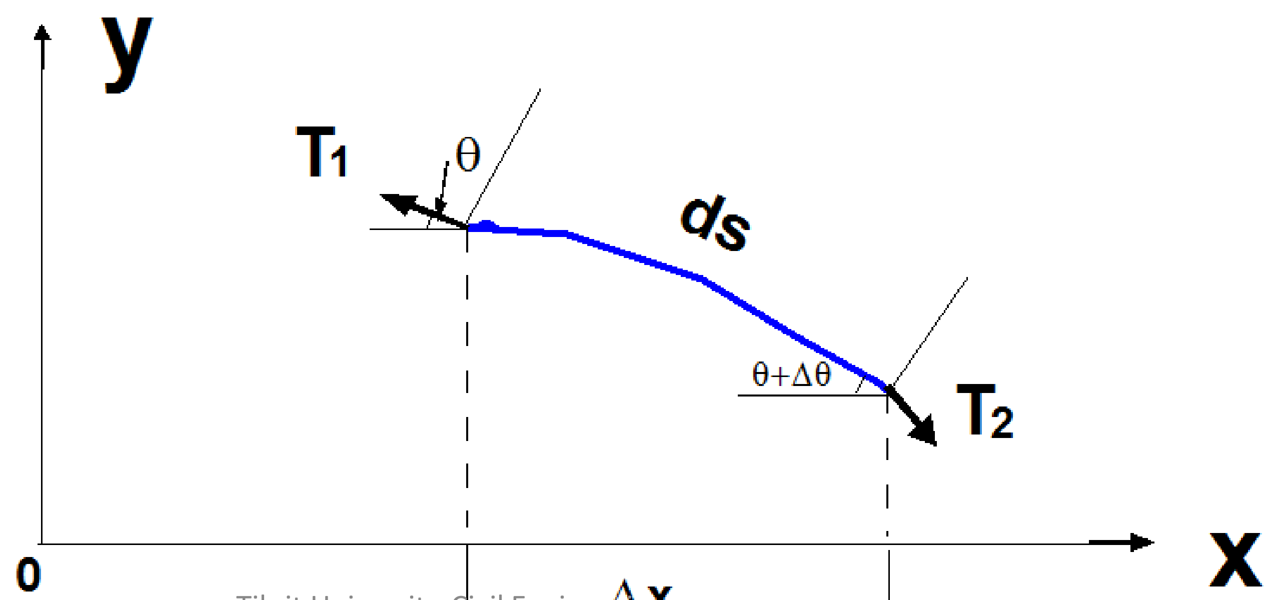
X


y
T1
θ
ds
θ+Δθ
T2
0
Δx
X

$$\sum Fx = 0$$

$$T_2 \cos(\theta + \Delta\theta) - T_1 \cos\theta = 0$$

$$\theta \; is \;\; too \;\; small \quad \theta \to 0$$

$$T_2 - T_1 = 0$$

$$\therefore T_2 = T_1 = T$$

$$\sum Fy = 0$$

$$T_1 \sin\,\theta - T_2 \sin(\theta + \Delta\theta) - \rho\,\Delta s(-\frac{\partial^2 y}{\partial t^2}) = 0$$

$$\rho = density \;\; per \;\; length$$

$$T_1 \sin \theta - T_2 \sin(\theta + \Delta\theta) - \rho \Delta s(-\frac{\partial^2 y}{\partial t^2}) = 0$$

$$\left.\begin{aligned} \sin\theta &= \theta \\ \sin(\theta + \Delta\theta) &= \theta + \Delta\theta \end{aligned}\right\} \theta \ \ is \ \ too \ \ small$$

$$T_1 = T_2 = T$$

$$T \ \theta - T \ (\theta + \Delta\theta) = -\rho \Delta s \frac{\partial^2 y}{\partial t^2}$$

$$\therefore T\Delta\theta = \rho \Delta s \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\Delta\theta}{\Delta s} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\Delta\theta}{\Delta s} = curvature$$

$$= \frac{\partial^2 y / \partial x^2}{[1 + (\frac{dy}{dx})^2]^{\frac{3}{2}}} \approx \frac{\partial^2 y}{\partial x^2}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\rho}{T} = a^2$$

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2} \qquad One-\text{dim}\,ensional\ \ wave\ Eq.$$

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2} \qquad (1)$$

By separation the variables :

$$y = X\ T$$

$$X = \text{function of } x$$

$$T = \text{function of } t$$

$$\therefore \left.\begin{array}{l} \dfrac{\partial^2 y}{\partial x^2} = X''T \\ \dfrac{\partial^2 y}{\partial t^2} = X\ T'' \end{array}\right\} \text{Subst. into Eq. (1)}$$

$$X''\ T = a^2\ X\ T''$$

$$X''\,T = a^2\,X\,T''$$

$$\frac{X''}{X} = a^2\,\frac{T''}{T}$$

$$\frac{X''}{a^2 X} = \frac{T''}{T} = Constant \rightarrow \begin{cases} -k^2 \\ 0 \\ k^2 \end{cases}$$

$$(1)\quad \frac{X''}{a^2 X} = \frac{T''}{T} = -k^2$$

$$X'' + a^2k^2X = 0 \Leftrightarrow \frac{d^2X}{dx^2} + a^2k^2X = 0$$

$$X = c_1 \cos kax + c_2 \sin kax$$

$$\frac{T''}{T} = -k^2$$

$$T'' + k^2 T = 0 \iff \frac{d^2T}{dt^2} + k^2 T = 0$$

$$T = c_3 \cos k\,t + c_4 \sin k\,t$$

$$\therefore y(x,t) = X.T$$

$$= (c_1 \cos k\,a\,x + c_2 \sin k\,a\,x)(c_3 \cos k\,t + c_4 \sin k\,t)$$

$$(2)\ Constant = 0$$

$$\frac{X''}{a^2X} = \frac{T''}{T} = 0$$

$$\frac{X''}{a^2X} = 0 \rightarrow X'' = 0 \rightarrow X' = c_1 \rightarrow X = c_1x + c_2$$

$$\frac{T''}{T} = 0 \rightarrow T'' = 0 \rightarrow T' = c_3 \rightarrow T = c_3t + c_4$$

$$\therefore y(x,t) = (c_1x + c_2)(c_3t + c_4)$$

$$(3)\; Constant = k^2$$

$$\frac{X''}{a^2 X} = \frac{T''}{T} = k^2$$

$$X'' + a^2 k^2 X = 0 \;\Leftrightarrow\; \frac{d^2 X}{dx^2} - a^2 k^2 X = 0$$

$$X = c_1 e^{kax} + c_2 e^{-kax}$$

$$\frac{T''}{T} = k^2$$

$$T'' + k^2 T = 0 \;\Leftrightarrow\; \frac{d^2 T}{dx^2} - k^2 T = 0$$

$$T = c_3 e^{kt} + c_4 e^{-kt}$$

$$\therefore y(x,t) = (c_1 e^{kax} + c_2 e^{-kax})(c_3 e^{kt} + c_4 e^{-kt})$$

For the string fixed at the both ends

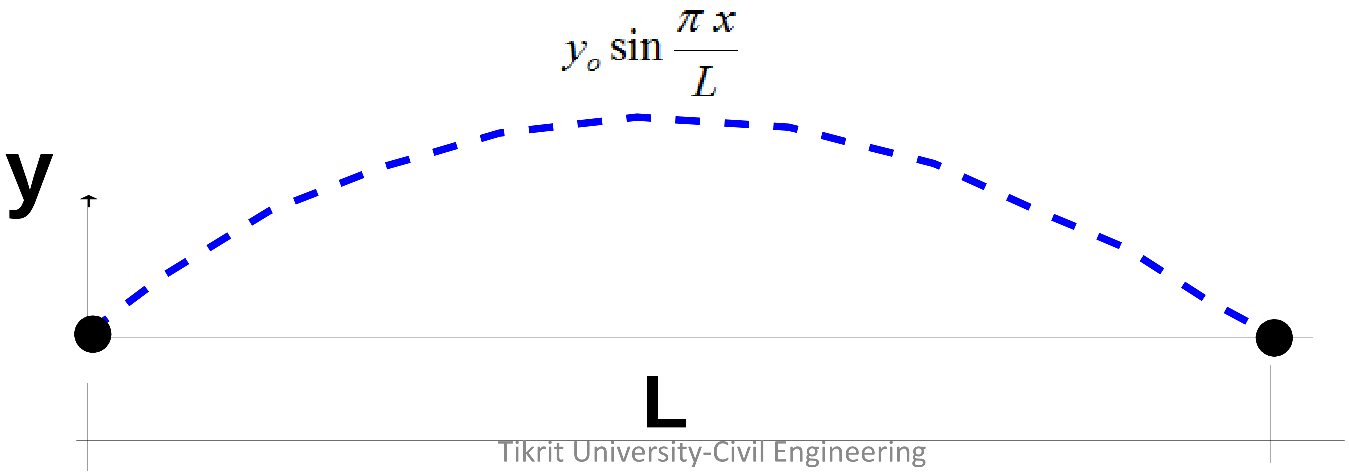
and subjected to the following :

Initial condition :

$$y(x,0) = y_o \sin\frac{\pi x}{L}$$

$$\left.\frac{\partial y}{\partial t}\right)(x,0) = 0$$

$$y(x,t) = (c_1 \cos k a x + c_2 \sin k a x)(c_3 \cos k t + c_4 \sin k t)$$



$$y(x,t) = (c_1 \cos k a x + c_2 \sin k a x)(c_3 \cos k t + c_4 \sin k t)$$

$$y(0,t) = 0 \quad \& \quad y(L,t) = 0$$

$$0 = (c_1 \cos 0 + c_2 \sin 0)(c_3 \cos k t + c_4 \sin k t)$$

$$c_1(c_3 \cos k t + c_4 \sin k t) = 0$$

$$Either; \quad c_1 = 0$$

$$Or; \quad (c_3 \cos k t + c_4 \sin k t) = 0$$

$$\therefore y(x,t) = c_2 \sin k a x(c_3 \cos k t + c_4 \sin k t)$$

$$Or; y(x,t) = \sin k a x(A \cos k t + B \sin k t)$$

$$Such \ \ that; \ c_2 * c_3 = A \quad \& \quad c_2 * c_4 = B$$

$$y(x,t)=\sin k\,a\,x(A\cos k\,t+B\sin k\,t)$$

$$y(L,t)=0$$

$$0=\sin k\,a\,L(A\cos k\,t+B\sin k\,t)$$

$$\sin k\,a\,L=0 \;\Rightarrow\; k\,a\,L=n\,\pi \;\Rightarrow\; \therefore k=\frac{n\,\pi}{a\,L}$$

$$y(x,t)=\sum_{n=1}^{\infty}\sin k\,a\,x(A_n\cos k\,t+B_n\sin k\,t)$$

$$y(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\,\pi\,x}{L}(A_n\cos\frac{n\,\pi}{a\,L}t+B_n\sin\frac{n\,\pi}{a\,L}t)$$

Initial conditions :

$$t = 0 \Rightarrow y(x,t) = y_o \sin\frac{\pi x}{L}$$

$$y_o \sin\frac{\pi x}{L} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}(A_n \cos 0 + B_n \sin 0)$$

$$y_o \sin\frac{\pi x}{L} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$A_n = \frac{2}{L}\int_0^L y_o \sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

$$A_n = \frac{2}{L}\,y_o\,L = 2\,y_o$$

$$\frac{\partial y}{\partial t}(x,0)=0$$

$$y(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi}{a\ L}t+B_n\sin\frac{n\pi}{a\ L}t\right)$$

$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(-\frac{n\pi}{a\ L}A_n\sin\frac{n\pi}{a\ L}t+\frac{n\pi}{a\ L}B_n\cos\frac{n\pi}{a\ L}t\right)$$

$$At\quad t=0\ ;\ \frac{\partial y}{\partial t}=0$$

$$0=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(-\frac{n\pi}{a\ L}A_n\sin 0+\frac{n\pi}{a\ L}B_n\cos 0\right)$$

$$0=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(\frac{n\pi}{a\ L}B_n\right)\ \Rightarrow\ \underline{B_n=0}$$

$$\therefore\ y(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\,2y_o\cos\frac{n\pi}{a\ L}t$$

$(2)\ One-\dim ensional\ \ consolidation:$

$$\frac{\partial^2 u}{\partial z^2} = a^2 \frac{\partial^2 u}{\partial t^2}$$

$$u(z,t) = Z_z \quad T_t$$

$$\frac{\partial^2 u}{\partial z^2} = Z''\ T$$

$$\frac{\partial u}{\partial t} = Z\ \ T'$$

$$Z''\ T = a^2 \quad Z\ \ T'$$

$$\frac{Z''}{a^2 Z} = \frac{T'}{T} = Cons\tan t = \begin{cases} -k^2 \\ \ \ 0 \\ \ \ k^2 \end{cases}$$

$$\frac{Z''}{a^2 Z} = -k^2$$

$$Z'' + a^2 k^2 Z = 0 \Leftrightarrow \frac{d^2 Z}{dx^2} + a^2 k^2 Z = 0$$

$$Z = c_1 \cos k\,a\,z + c_2 \sin k\,a\,z$$

$$\frac{T'}{T} = -k^2$$

$$T' + k^2 T = 0 \Leftrightarrow \frac{dT}{dt} + k^2 T = 0$$

$$T = c_3 e^{-k^2 t}$$

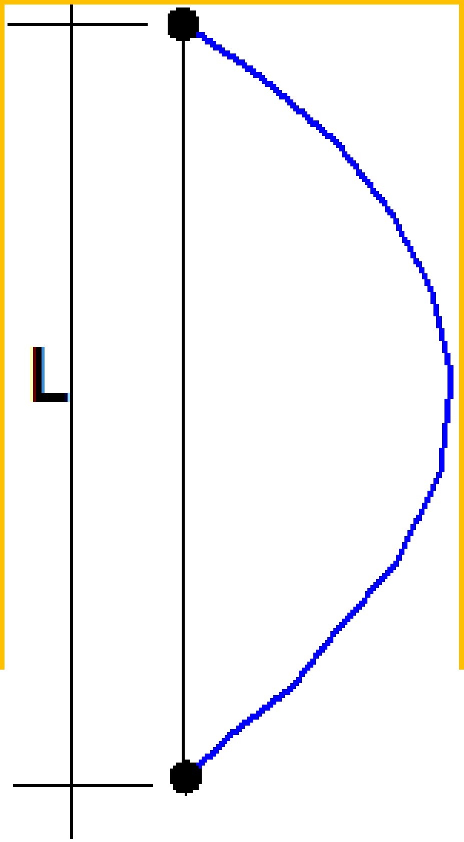
$$\therefore u(z,t) = Z.T$$

$$= (c_1 \cos k\,a\,z + c_2 \sin k\,a\,z)\; c_3\, e^{-k^2 t}$$

$$u(z,t) = c_3 e^{-k^2 t}(c_1 \cos k a z + c_2 \sin k a z)$$

$$Let \quad A = c_1 * c_2 \quad \& \quad B = c_1 * c_3$$

$$\therefore \underline{\underline{u(z,t) = e^{-k^2 t}(A \cos k a z + B \sin k a z)}} \quad general \ \ Sol.$$



<u>Boundary Conditions :</u>

$$u(0,t) = 0$$

$$u(L,t) = 0$$

<u>Initial Condition :</u>

$$u(z,0) = f(z) \ given$$

A clay layer :

$$u(z,0) = z(L - z)$$

$$u(z,t) = e^{-k^2 t}(A\cos k\,a\,z + B\sin k\,a\,z)$$

$$(1)\,u(o,t) = 0$$

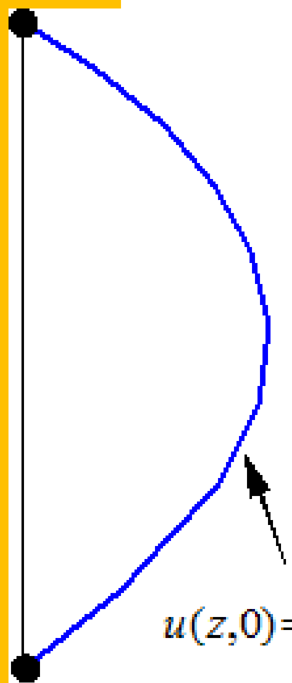
$$0 = e^{-k^2 t}(A\cos 0 + B\sin 0)$$

$$0 = e^{-k^2 t}A \quad \Rightarrow \quad A = 0$$

$$(1)\,u(L,t) = 0$$

$$0 = e^{-k^2 t} \quad B\sin k\,a\,L$$

$$B \neq 0$$



$$\therefore \sin k\,a\,L = 0 \quad \Rightarrow \quad k\,a\,L = n\pi \Rightarrow k = \frac{n\pi}{a\,L}$$

$$u(z,t)=\sum_{n=1}^{\infty} B \sin k a z\, e^{-k^2 t}$$

$$=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} z\ \ e^{-k^2 t}$$

$$u(z,0)=z(L-z)$$

$$z(L-z)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} z\ \ e^{0}$$

$$z(L-z)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} z$$

$$Bn=\frac{2}{L}\int_0^L z(L-z)\sin\frac{n\pi z}{L} dz$$

$$\vdots$$

$$\vdots$$

$H.\,W.$

Solution of Partial Differential Equations:

Here under we consider the simple examples, the solution of which depends up to the meaning of partial differentiation.

Example (1):

Solve; $(i)\ \dfrac{\partial z}{\partial x}=0$

$$(ii)\ \frac{\partial^2 z}{\partial y^2}=0$$

$$(iii)\ \frac{\partial^3 z}{\partial x^2 \partial y}=0$$

Noting that $\underline{z}$ *is a function of* $\underline{x\ \&\ y}$.

Solution :

$$(i)\ \frac{\partial z}{\partial x} = 0 \qquad (a)$$

Here z is a function of two independent variables x and y. On integrating Eq. (a), we get that solution as;

$z =$ *function independent of* x.

$\therefore z = \phi(y)$ *which is arbitrary.*

$$(ii)\ \frac{\partial^2 z}{\partial y^2}=0$$

z *is a function* x *and* y.

Integrating with respect to y, *we get;*

$$\frac{\partial z}{\partial y}=a \text{ function independent of } y$$

$$=\phi(x)\ \ (\ldots\text{the other variable is } x \text{ only})$$

Again integrating with respect to y, *we get;*

$$z=\int \phi(x)\,dy + a \text{ function independent of } y$$

$$=\phi(x)\int dy + f(x)$$

$$\therefore z=\phi(x)\,y + f(x) \text{ where } \phi(x) \text{ and } f(x) \text{ are arbitrary in the solution}$$

$$(iii)\ \frac{\partial^3 z}{\partial x^2 \partial y} = 0$$

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial}{\partial y}\{\frac{\partial^2 z}{\partial x^2}\} = 0$$

$$On\ \text{int}\,egrating\ \ with\ respect\ to\ y,\ we\ get;$$

$$\frac{\partial^2 z}{\partial x^2} = a\ function\ independen\,t\ of\ \ y$$

$$= \phi(x) \qquad\qquad\qquad\qquad (say)$$

$$Now;\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\{\frac{\partial z}{\partial x}\} = \phi(x)$$

$$Now;\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\{\frac{\partial z}{\partial x}\} = \phi(x)$$

$On\ integrating\ \ with\ respect\ to\ x,\ we\ get;$

$$\frac{\partial z}{\partial x} = \int \phi(x)\,dx + a\ function\ independent\ of\ \ x$$

$$= \int \phi(x)dx + q(y)$$

$Again\ \ integrating\ \ with\ respect\ to\ x,\ we\ get;$

$$\therefore z = [\int \phi(x)dx]\,dx + \int q(y)\,dx + F(y)$$

$\underline{\underline{Or:}}$

$$z = \iint \phi(x)dxdx + \ x\,q(y) + F(y)\ which\ is\ required\ \ solution$$

Example (2): *Solve the partial differential equations in the following case;*

$$(i)\ \frac{\partial^2 z}{\partial x^2} = \sin x$$

$$(ii)\ \frac{\partial^2 u}{\partial x\,\partial t} = e^{-t}\cos x \quad \text{given that } u = 0 \text{ when } t = 0$$

$$\text{and } \frac{\partial u}{\partial t} = 0 \text{ when } x = 0$$

$$(iii)\ \frac{\partial^2 z}{\partial x\partial y} = \sin x \ \sin y \quad \text{for which } \frac{\partial z}{\partial y} = -2\sin y \text{ when } x = 0$$

$$\text{and } z = 0 \text{ when } y \text{ is an odd multiply of } \frac{\pi}{2}$$

$$(iv)\ \frac{\partial^2 z}{\partial x\partial y} = \frac{1}{x+y} \quad \text{given that } z = y\ln y \text{ and}$$

$$\frac{\partial z}{\partial y} = 1 + \ln y \quad \text{when } x = 0$$

Solution :

$$(i)\ \frac{\partial^2 z}{\partial x^2} = \sin x$$

Integrating w.r.t. x, we get;

$$\frac{\partial z}{\partial x} = \int \sin x\, dx + f(y) = -\cos x + f(y)$$

Again integrating w.r.t. x, we get;

$$z = -\int \cos x\, dx + \int f(y)\, dx + g(y)$$

$$= -\sin x + x f(y) + g(y)$$

where $f(y)$ and $g(y)$ are arbitrary functions.

$$(ii)\ \frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \qquad \begin{cases} u = 0 & when \quad t = 0 \\ \dfrac{\partial u}{\partial t} = 0 & when \quad x = 0 \end{cases}$$

Integrating partially w.r.t. x, we get;

$$\frac{\partial u}{\partial t} = \int e^{-t} \cos x \, dx + a \ function \ independent \ of \ (x)$$

$$= e^{-t} \int \cos x \, dx + f(t) \qquad (say)$$

$$\therefore \frac{\partial u}{\partial t} = e^{-t} \sin x + f(t)$$

$$x = 0 \Rightarrow \frac{\partial u}{\partial t} = 0$$

$$0 = e^{-t} \sin 0 + f(t) \Rightarrow f(t) = 0$$

$$\therefore \frac{\partial u}{\partial t} = e^{-t} \sin x$$

$$\frac{\partial u}{\partial t} = e^{-t}\sin x$$

$$Now\ \text{int}\,egrating\ \ partially\ \ w.r.t.\ t,\ we\ get;$$

$$u(x,t) = \int e^{-t}\sin x\,dt + a\ function\ \ independent\ of\ \ (t)$$

$$= \sin x\int e^{-t}\,dt + g(x) \qquad (say)$$

$$u(x,t) = -e^{-t}\sin x + g(x)$$

$$t = 0 \Rightarrow u = 0$$

$$0 = -e^{0}\sin x + g(x) \Rightarrow \ f(x) = \sin x$$

$$\therefore u(x,t) = -e^{-t}\sin x + \sin x = \underline{\underline{\sin x(1 - e^{-t})}}$$

$$(iii)\ \frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y \qquad \textit{for which } \frac{\partial z}{\partial y} = -2\sin y \textit{ when } x = 0$$

$$\textit{and } z = 0 \textit{ when } y \textit{ is an odd multiply of } \frac{\pi}{2}$$

Integrating partially w.r.t. x, we get;

$$\frac{\partial z}{\partial y} = \sin y \int \sin x\, dx + f(y)$$

$$= \sin y(-\cos x) + f(y)$$

$$x = 0 \Rightarrow \frac{\partial z}{\partial y} = -2\sin y$$

$$-2\sin y = \sin y(-\cos 0) + f(y)$$

$$-2\sin y = \sin y(-1) + f(y) \Rightarrow f(y) = -\sin y$$

$$\therefore \frac{\partial z}{\partial y} = -\sin y \cos x - \sin y = -\sin y(\cos x + 1)$$

$$\frac{\partial z}{\partial y} = -(\cos x + 1)\sin y$$

$$Now \;\; integrating \;\; partially \;\; w.r.t. \; y, \; we \; get;$$

$$z = -(\cos x + 1)\int \sin y \, dy + g(x)$$

$$= (\cos x + 1)\cos y + g(x)$$

$$z = 0 \Rightarrow \quad y = odd * \frac{\pi}{2}$$

$$0 = (\cos x + 1)(0) + g(x) \;\Rightarrow\; g(x) = 0$$

$$\underline{\underline{\therefore z = (\cos x + 1)\cos y}}$$

$$(iv)\ \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x+y} \qquad \textit{given that } z = y\ln y \textit{ and}$$

$$\frac{\partial z}{\partial y} = 1 + \ln y \quad \textit{when} \quad x = 0$$

Integrating partially w.r.t. x, we get;

$$\frac{\partial z}{\partial y} = \int \frac{1}{x+y}\,dx + f(y)$$

$$= \ln(x+y) + f(y)$$

$$x = 0 \Rightarrow \frac{\partial z}{\partial y} = 1 + \ln y$$

$$1 + \ln y = \ln(0+y) + f(y)$$

$$\therefore\ f(y) = 1 + \ln y - \ln y = 1$$

$$\therefore\ \frac{\partial z}{\partial y} = \ln(x+y) + 1$$

$$\frac{\partial z}{\partial y} = \ln(x+y) + 1$$

$$Now \ \ \text{int}\,egrating \ \ partially \ \ w.r.t. \ y, we \ get;$$

$$z = \int \ln(x+y)\,dy + \int 1 dy + g(x)$$

$$\int u dv = uv - \int v du \qquad \{\ln(x+y) = u \quad and \ \ dy = dv)$$

$$z = \ln(x+y)\,y - \int \left(\frac{1}{x+y}y\right) dy + y + g(x)$$

$$= y\ln(x+y) - \int \left(1 - \frac{x}{x+y}\right) dy + y + g(x)$$

$$= y\ln(x+y) - \int 1 dy + x\int \frac{dy}{x+y} + y + g(x)$$

$$z = y\ln(x+y) - \int 1dy + x\int \frac{dy}{x+y} + y + g(x)$$

$$= y\ln(x+y) - y + x\ln(x+y) + y + g(x)$$

$$= y\ln(x+y) + x\ln(x+y) + g(x)$$

$$\therefore z(x,y) = (x+y)\ln(x+y) + g(x)$$

$$x = 0 \quad \Rightarrow z = y\ln y$$

$$y\ln y = (0+y)\ln(0+y) + g(x) \Rightarrow g(x) = 0$$

$$\underline{\underline{\therefore z = (x+y)\ln(x+y)}}$$

Numerical Solution of Partial Differential Equations:

The general second-order linear partial differential equation is of the form:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

Which can be written as:

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G \qquad ....(1)$$

Where $A, B, C, \ldots\ldots, G$ are all functions of x & y

$$au_{xx} + 2bu_{xy} + cu_{yy} = F(x, y, u, u_x, u_y)$$

u *is the unknown function.*

Equations of form (1) *can be classified with respect to the sign of the discriminant:*

$$\Delta s = B^2 - 4AC$$

in the following ways:

(1) *If* $\underline{\Delta s < 0}$ *at a point in the* (x, y) *plane, the equation is said to be* *__elliptic type__.*

(2) *If* $\underline{\Delta s > 0}$ *at that point is said to be* *__hyperbolic type__.*

(3) *__Parabolic type__* *when* $\underline{\Delta s = 0}$.

$Elliptic\ \ type \qquad 4ac - b^2 > 0 \qquad Laplace\ \ equation$

$Parabolic\ type \qquad 4ac - b^2 = 0 \qquad Heat\ \ equation$

$Heperbolic\ \ type \qquad 4ac - b^2 < 0 \qquad Wave\ equation$

$In\ the\ \ following\,,\, we\ will\ \ restrict\ our\ \ solves\ to\ three$

$simple\ \ particular\ \ cases\ \ of\ \ \ Eq.\,(1);\, namely:$

$$u_{xx} + u_{yy} = 0 \qquad (the\ Laplace\ \ equation)$$

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0 \qquad (the\ wave\ \ equation)$$

$$u_{xx} - u_t = 0 \qquad (the\ heat\ \ conduction\ \ equation)$$

Finite – Difference Approximations to derivatives:

Let the (x, y) *plane be divided into a network of rectangles of sides* $\underline{\Delta x = h}$ *and* $\underline{\Delta y = k}$ *by drawing the sets of lines:*

$$x = ih; \qquad i = 0, 1, 2, 3, \ldots\ldots$$

$$y = jk; \qquad j = 0, 1, 2, 3, \ldots\ldots$$

The points of intersection of these families of lines are called ***mesh points***, ***lattice points*** *or* ***grid points***.

Similarly we have the approximations :

$$u_y = \frac{u_{i,\,j+1} - u_{i,\,j}}{k} + 0(k) \qquad \textit{Forward differenc}$$

$$u_y = \frac{u_{i,\,j} - u_{i-1,\,j}}{h} + 0(h) \qquad \textit{Backward differenc}$$

$$u_x = \frac{u_{i+1,\,j} - u_{i-1,\,j}}{2h} + 0(h^2) \qquad \textit{Central differenc}$$

and;

$$u_{xx} = \frac{u_{i-1,\,j} - 2u_{i,\,j} + u_{i+1,\,j}}{h^2} + 0(h^2)$$

Where $u_{i,\,j} = u(ih,\, jk) = u(x, y)$

Similarly we have the approximations :

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + 0(k) \qquad \textit{Forward differenc}$$

$$u_y = \frac{u_{i,j} - u_{i,j-1}}{k} + 0(k) \qquad \textit{Backward differenc}$$

$$u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + 0(k^2) \qquad \textit{Central differenc}$$

and;

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + 0(k^2)$$

We can now obtain the finite-difference analogues of partial differential equations by replacing the derivatives in any equation by their corresponding difference approximation given above.

$Thus,\ the\ Laplace\ equation\ in\ two\ \dim ension, namely;$

$$u_{xx} + u_{yy} = 0$$

$has\ its\ finite - difference\ ana\log ue;$

$$\frac{1}{h^2}[u_{i+1,\,j} - 2u_{i,\,j} + u_{i-1,\,j}] + \frac{1}{k^2}[u_{i,\,j+1} - 2u_{i,\,j} + u_{i,\,j-1}] = 0$$

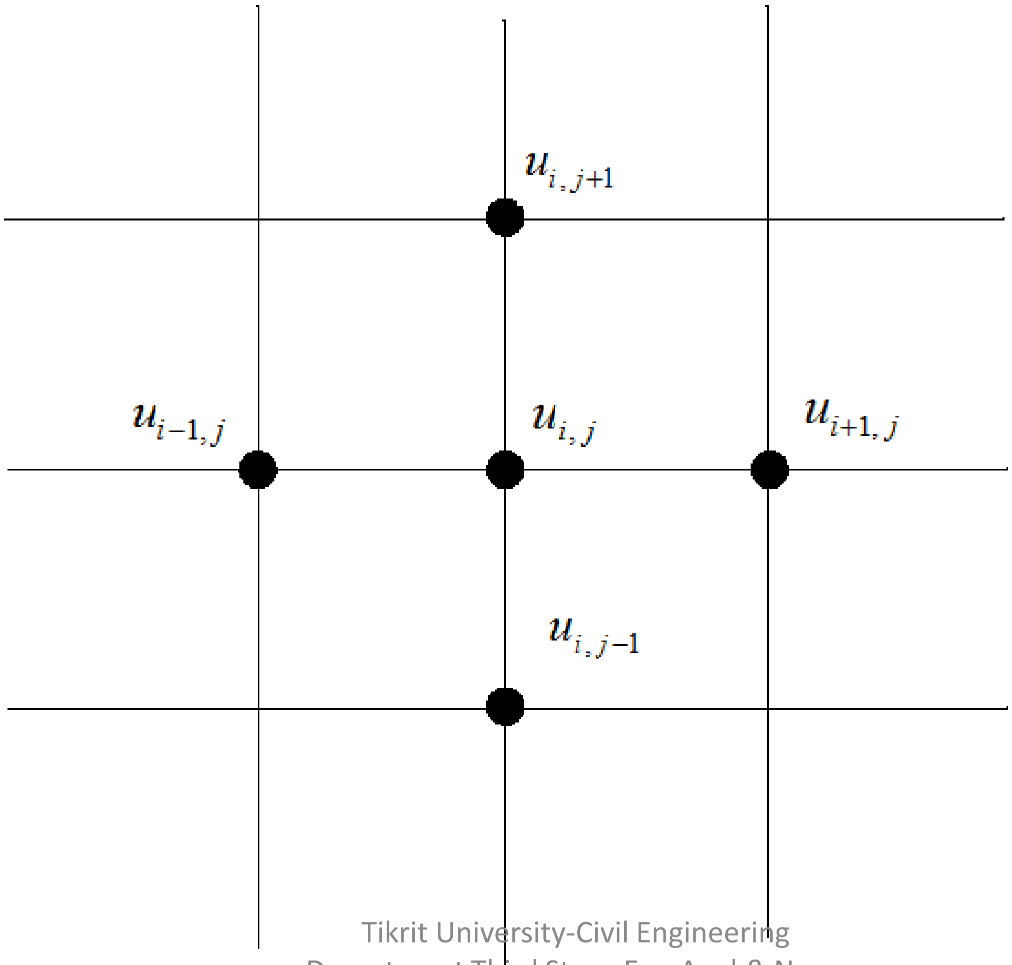
$If\quad h = k,\ this\ gives;$

$$u_{i,\,j} = \frac{1}{4}[u_{i+1,\,j} + u_{i-1,\,j} + u_{i,\,j+1} + u_{i,\,j-1}] \qquad (a)$$

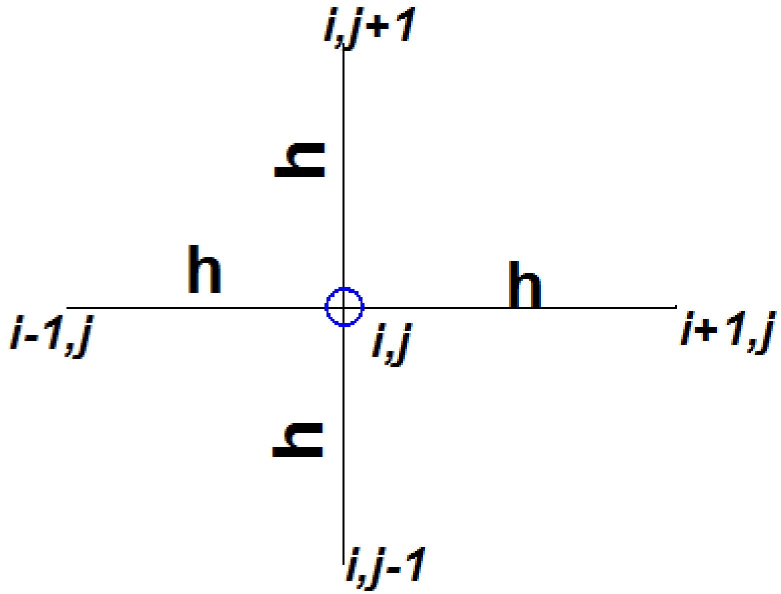
$Which\ shows\ that\ the\ value\ of\ u\ at\ any\ po\text{int}\ is\ the\ mean$ $of\ its\ values\ at\ the\ \underline{four\ neighbouring\ po\text{int}s}. This\ is\ called$ $the\ \underline{s\tan dard\ five - po\text{int}\ formula}.$

$$The\ \underline{standard\ five-point\ formula}\ is\ written:$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$



Dirichlet problem



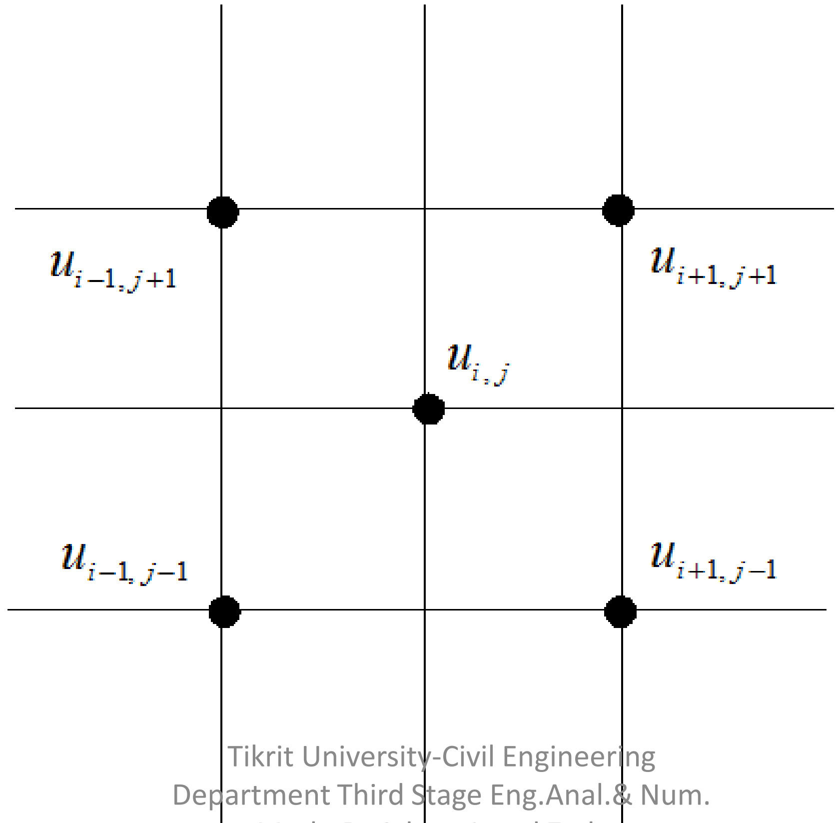
$$\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix}$$

Also, instead of formula (a), we may use the following formula:

$$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}]$$

Which uses the function values at the diagonal points.

And is therefore called the diagonal five-point formula.

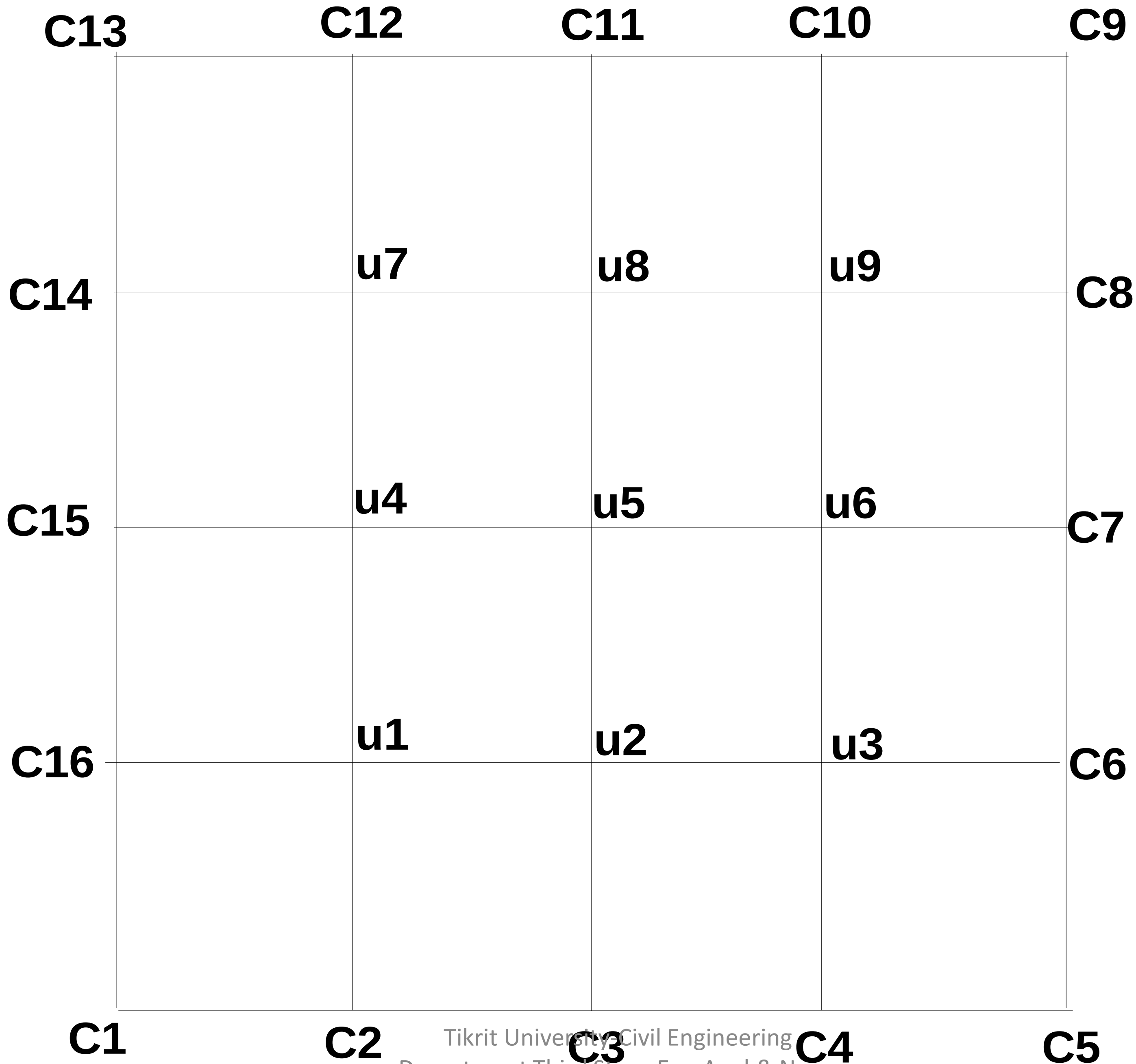


Laplace's equation:

We wish to solve Laplace's equation:

$$u_{xx}+u_{yy}=0$$

in a boundary region *R* with boundary *C*. As in Dirichlet's problem, let the value of *u* be specified every where on *C*. For simplicity, let *R* be a square region so that it can be divided into a network of small squares of side *h*. Let the values of *u(x, y)* on the boundary *C* be given by *Ci* and let the interior mesh points and the boundary points be as in the figure below:


C13
C12
C11
C10
C9
u7
u8
u9
C14
C8
u4
u5
u6
C15
C7
u1
u2
u3
C16
C6
C1
C2
C3
C4
C5

$We\ \ first\ use\ \ the\ diagonal\ \ five-point\ \ formula\ and$

$compute;\ u_5, u_7, u_9, u_1\ \ and\ \ u_3\ in\ this\ \ order,\ Thus\ we\ obtain;$

$$u_5 = \frac{1}{4}[C_1 + C_5 + C_9 + C_{13}];$$

$$u_7 = \frac{1}{4}[C_{15} + u_5 + C_{11} + C_{13}];$$

$$u_9 = \frac{1}{4}[u_5 + C_7 + C_9 + C_{11}];$$

$$u_1 = \frac{1}{4}[C_1 + C_3 + u_5 + C_{15}]\ and$$

$$u_3 = \frac{1}{4}[C_3 + C_5 + C_7 + u_5]$$

We then compute, in this order, the remaining quantities, such u_8, u_4, u_6 and u_2 by the standard five-point formula. Thus we have;

$$u_8 = \frac{1}{4}[u_5 + u_9 + C_{11} + C_7];$$

$$u_4 = \frac{1}{4}[u_1 + u_5 + u_7 + C_{15}];$$

$$u_6 = \frac{1}{4}[u_5 + u_3 + C_7 + u_9] \text{ and}$$

$$u_2 = \frac{1}{4}[C_3 + u_3 + u_5 + u_1]$$

$When\ \ once\ \ all\ \ the\ u_i (i = 1,2,3,......9)\ are\ computed,$

$their\ \ accuracy\ can\ be\ \ improved\ \ by\ any\ of\ \ the$

$iterative\ methods\ \ described\ below:$

<u>$(1)\ Jacobi's\ method:$</u>

$Let\ u_{i,j}^{(n)}\ \ denote\ the\ n^{th}\ \ iterative\ \ \ value\ of\ \ u_{i,j}.$

$An\ iterative\ \ procedure\ to\ solve\ the\ Eq.(a)\ is:$

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}]$$

$for\ \ the\ interior\ \ \ mesh\ \ points.\ This\ \ is\ \ called\ \ the$

$point\ \ Jacobi\ \ method.$

(2) Gauss – Seidal method :

The method uses the latest iterative values available and scans the mesh points systematically from left to right along successive rows.

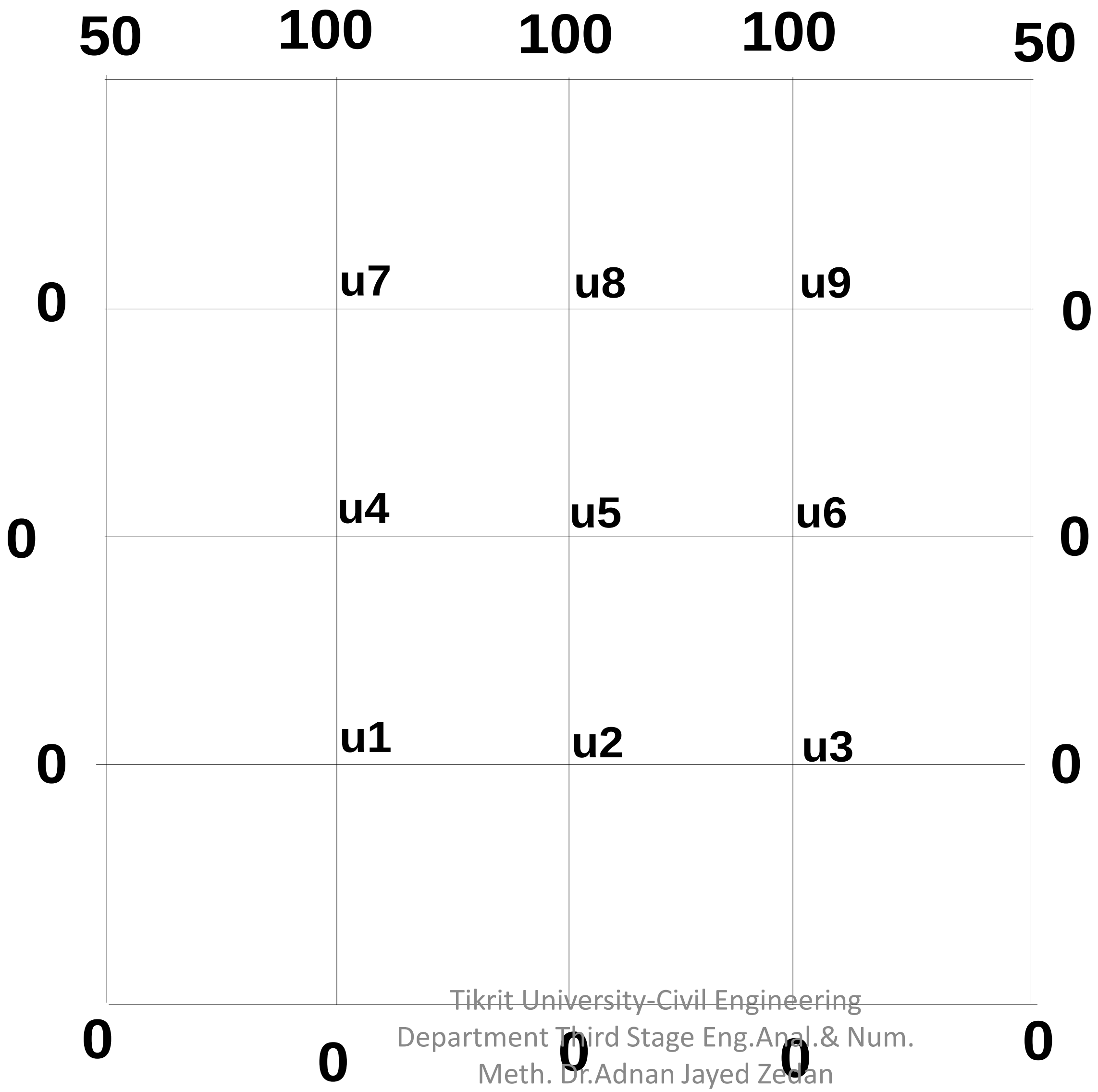
The iterative frmula is :

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}]$$

It can be shown that the Gauss – Seidal scheme converges twice as fast as the Jacobi scheme.

Example:

Solve Laplace equation for the figure given below:



$\underline{Solution\ :}$

$We\ \ first\ compute\ \ the\ \ \ quantities\,;u_5,u_7,u_9,u_1\ \ \ and\ \ u_3$

$by\ u\sin g\ \ the\ diagonal\ \ five - po\text{int}\ \ formula:$

$$u_5^{(1)} = \frac{1}{4}[0+0+50+50] = 25$$

$$u_7^{(1)} = \frac{1}{4}[0+25+100+50] = 43.75$$

$$u_9^{(1)} = \frac{1}{4}[25+0+50+100] = 43.75$$

$$u_1^{(1)} = \frac{1}{4}[0+0+25+0] = 6.25$$

$$u_3^{(1)} = \frac{1}{4}[0+0+0+25] = 6.25$$

$We\ now\ compute;\ u_8, u_4, u_6\ \ and\ u_2\ successively$
$by\ using\ the\ standard\ five-point\ formula:$

$$u_8^{(1)} = \frac{1}{4}[25 + 43.75 + 100 + 43.75] = 53.125$$

$$u_4^{(1)} = \frac{1}{4}[0 + 6.25 + 25 + 43.75] = 18.75$$

$$u_6^{(1)} = \frac{1}{4}[25 + 6.25 + 0 + 43.75] = 18.75$$

$$u_2^{(1)} = \frac{1}{4}[6.25 + 0 + 6.25 + 25] = 9.375$$

We have thus obtained the first approximations of all the nine mesh points and we can now use one of the aterative formula, by using Gauss – Seidal formula:

$$u_{i,j}^{(n+1)} = \frac{1}{4}[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}]$$

n	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
1	6.25	9.38	6.25	18.75	25.00	18.75	43.75	53.13	43.75
2	7.03	9.57	7.08	18.94	25.10	18.98	43.02	52.97	42.99
3	7.13	9.83	7.20	18.81	25.15	18.84	42.94	52.77	42.90
4	7.16	9.88	7.18	18.81	25.08	18.79	42.89	52.72	42.88
5	7.17	9.86	7.16	18.78	25.04	18.77	42.88	52.70	42.87

$$u_{1,1}^{(2)} = \frac{1}{4}[u_{0,1}^{(2)} + u_{2,1}^{(1)} + u_{1,0}^{(2)} + u_{1,2}^{(1)}]$$

$$u_1 = \frac{1}{4}[0 + 9.38 + 0 + 18.75] = 7.03$$

$$u_{2,1}^{(2)} = \frac{1}{4}[u_{1,1}^{(2)} + u_{3,1}^{(1)} + u_{2,0}^{(2)} + u_{2,2}^{(1)}]$$

$$u_2 = \frac{1}{4}[7.03 + 6.25 + 0 + 25] = 9.57$$