Linear Differential Equations with Constant Coefficients

The general linear second-order equation:

$$y'' + P(x)y' + Q(x)y = R(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$y'' + P(x)y' + Q(x)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$P(x), Q(x)$ and $Q(x)$ are known functions.

Eq.(1) is linear, 2^{nd} order and Non homog.

Eq.(2) is linear, 2^{nd} order and Homog.

$Theorem\,1:$

$If\ \ y_1\ and\ y_2\ are\ any\ solutions\ of\ the$

$\hom ogeneous\ equation:$

$$y'' + P(x)y' + Q(x)y = 0,$$

$then\ \ y_3 = c_1y_1 + c_2y_2,\ where\ c_1\ and\ c_2\ are$

$arbitrary\ \ cons\tan ts,\ is\ also\ solution.$

$Theorem\,2:$

$If\ \ y_1\ and\ y_2\ are\ two\ solutions\ of\ the\ homogeneous$

$equation: y'' + P(x)y' + Q(x)y = 0, \quad for\ which$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \neq 0 \quad and\ if$$

$\int P(x)dx\ exists,\ then\ there\ exist\ constants\ c_1\ \&\ c_2$

$such\ that\ any\ solution\ \ y_3\ of\ the\ homogeneous$

$equation\ can\ be\ expressed\ in\ the\ form$

$$y_3 = c_1 y_1 + c_2 y_2$$

Proof :

To prove this theorem, first show that any pair of solution of : $y'' + P(x)y' + Q(x)y = 0$, *say* y_i *and* y_j *satisfies the relation :*

$$W(y_i, y_j) = y_i y_j' - y_j y_i' = k_{ij} e^{-\int P(x)dx}$$

where k_{ij} *is a suitable constant.*

$$y_i'' + P(x)y_i' + Q(x)y_i = 0 \quad ....(1)$$

$$y_j'' + P(x)y_j' + Q(x)y_j = 0 \quad ....(2)$$

$$[\,Eq.(2) \times y_i\,] - [\,Eq.(1) \times y_i\,] \;\; get;$$

$$[y_i y_j'' - y_j y_i''] + P(x)[y_i y_j' - y_i y_j'] = 0 \;\; ..(3)$$

$Now:$

$$\frac{dW(y_i, y_j)}{dx} = \frac{d(y_i y_j' - y_j y_i')}{dx}$$

$$= (y_i y_j'' + y_j' y_i') - (y_j y_i'' + y_i' y_j')$$

$$= (y_i y_j'' - y_j y_i'')$$

$$Hence,\ Eq.(3)\ can\ be\ written;$$

$$\frac{dW(y_i, y_j)}{dx} + P(x)\,W(y_i, y_j) = 0$$

$$\ln W(y_i, y_j) = -\int P(x)dx + c$$

$$W(y_i, y_j) = k_{ij}\, e^{-\int P(x)dx}$$

$Now,\ for\ two\ pairs\ of\ solutions$

$(y_3, y_1)\ and\ (y_3, y_2)$

$$y_3 y_1' - y_1 y_3' = k_{31} e^{-\int P(x)dx}$$

$$y_3 y_2' - y_2 y_3' = k_{32} e^{-\int P(x)dx}$$

$Solve,\ these\ two\ above\ equations,$

$getting;$

$$y_3 = \frac{y_1 k_{32} e^{-\int P(x)dx} - y_2 k_{31} e^{-\int P(x)dx}}{y_1 y_2' - y_2 y_1'}$$

$$y_3 = \frac{y_1 k_{32} e^{-\int P(x)dx} - y_2 k_{31} e^{-\int P(x)dx}}{k_{12} e^{-\int P(x)dx}}$$

$$\therefore y_3 = \frac{k_{32}}{k_{12}} y_1 - \frac{k_{31}}{k_{12}} y_2$$

$$y_3 = c_1 y_1 + c_2 y_2$$

$$where \quad c_1 = \frac{k_{32}}{k_{12}} \quad and \;\; c_2 = -\frac{k_{31}}{k_{12}}$$

$Let\ us\ \sup pose\ that\ \ y_1(x) \neq 0\ is\ a\ solution\ of,$

$y'' + P(x)y' + Q(x)y = 0,\ and\ let\ us\ attempt\ to$

$find\ a\ function\ \Phi(x)\ with\ the\ property\ that$

$\Phi(x)\,y_1(x)\ is\ also\ a\ solution\ of\ above\ equation.$

$Now,\ substitute\ \ y = \Phi(x)y_1(x)\ \ \mathrm{int}\,o\ above\ equation:$

$$(y_1''\Phi + 2y_1'\Phi' + y_1'\Phi'') + P(x)(y_1'\Phi + y_1\Phi') + Q(x)(y_1\Phi)$$

$$= (y_1'' + P(x)y_1' + Q(x)y_1)\Phi + (2y_1' + P(x)y_1)\Phi' + y_1\Phi'' = 0$$

$$y_1\Phi'' + (2y_1' + P(x)y_1)\Phi' = 0$$

$$\frac{d\Phi'}{\Phi'} + [\frac{2y_1'}{y_1} + P(x)]dx = 0$$

$$\ln(\Phi') + 2\ln(y_1) + \int P(x)dx = \ln(c)$$

$$\ln\frac{\Phi' y_1^2}{c} = -\int P(x)dx$$

$$\ln \frac{\Phi' y_1^2}{c} = -\int P(x)dx$$

$$\Phi' = \frac{c\, e^{-\int P(x)dx}}{y_1^2}$$

$$\Phi = c\int \frac{e^{-\int P(x)dx}}{y_1^2} dx + k$$

For all values of $c\,\&\,k$, *the solution;*

$$\Phi(x)\, y_1(x) = c\, y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2} dx + k\, y_1(x)$$

$$\Phi(x)\, y_1(x) = c\, y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2} dx + k\, y_1(x)$$

$The\ \Phi(x) y_1(x)\ is\ also\ a\ complete\ solution,$

$\sin ce\ 2\ cons\tan ts\ provided$

$W(y_1, \Phi y_1) \neq 0$

Example :

Find a complete solution of equation;

$x^2 y'' + xy' - 4y = 0,$ *given that* $y = x^2$ *is one solution*

Solution :

Substitute $y = x^2\Phi$ *into the given D.E.*

$$x^2(x^2\Phi)'' + x(x^2\Phi)' - 4(x^2\Phi) = 0$$

$$x^2(2\Phi + 2x\Phi' + 2x\Phi' + x^2\Phi'') + x(2x\Phi + x^2\Phi') - 4(x^2\Phi) = 0$$

$$2x^2\Phi + 4x^3\Phi' + x^4\Phi'' + 2x^2\Phi + x^3\Phi' - 4x^2\Phi = 0$$

$$x^3(x\Phi'' + 5\Phi') = 0$$

$$x\Phi'' + 5\Phi' = 0$$

$$x\frac{d\Phi'}{dx}+5\Phi'=0$$

$$\frac{d\Phi'}{\Phi'}+5\frac{dx}{x}=0$$

$$\ln\Phi'+5\ln x=\ln c$$

$$\Phi'=\frac{c}{x^5}$$

$$\Phi=-\frac{c}{4x^4}+k$$

$$The\ complete\ \ solution\ \ is:$$

$$y=x^2\Phi=-\frac{c}{4x^2}+kx^2$$

Theorem 3:

If Y *is any solutions of the non*homogeneous *equation*:

$$y'' + P(x)y' + Q(x)y = R(x),$$

and if $c_1y_1 + c_2y_2$ *is a complete solution of the* homogeneous *equation obtained from this by deleting the term* $R(x)$, *then*

$y = c_1y_1 + y_2c_2 + Y$ *is a complete solution of the non*homogeneous *equation*.

Proof :

Let y_i *be any solution of nonhomogeneous equation* :

$$y_i'' + P(x)y_i' + Q(x)y_i = R(x) \quad ....(1)$$

and similarly, since Y *is a solution of nonhomogeneous equation* :

$$Y'' + P(x)Y' + Q(x)Y = R(x) \quad ....(2)$$

$Eq.(1) - Eq.(2);$

$$(y_i'' - Y'') + P(x)(y_i' - Y') + Q(x)(y_i - Y) = 0$$

OR;

$$(y_i - Y)'' + P(x)(y_i - Y)' + Q(x)(y_i - Y) = 0$$

Thus the quantity $(y_i - Y)$ *satisfies the homogeneous equation and, hence, by theorem* (2), *must be expressible in the form*;

$$y_i - Y = c_1 y_1 + c_2 y_2$$

Since y_i *was any solution of the nonhomogeneous equation. Theorem* (3) *is thus established.*

The homogeneous linear equation with constant coefficients:

$$\frac{dy}{dx} = Dy, \qquad \frac{d^2y}{dx^2} = D(Dy) = D^2y, \qquad \frac{d^3y}{dx^3} = D^3y$$

$$(aD^2 + bD + c)\phi(x) \text{ means } aD^2\phi(x) + bD\phi(x) + c\phi(x)$$

$$(3D^2 - 10D - 8)x^2 = 3(2) - 10(2x) - 8x^2 = 6 - 20x - 8x^2$$

$$(3D+2)(D-4)x^2 = (3D+2)(2x-4x^2) = (6-24x) + (4x-8x^2)$$
$$= 6 - 20x - 8x^2$$

$$(D-4)(3D+2)x^2 = (D-4)(6x+2x^2) = (6+4x) - (24x+8x^2)$$
$$= 6 - 20x - 8x^2$$

$ay'' + by' + cy = 0$ can be written $(aD^2 + bD + c)y = 0$

It natural to try: $y = e^{mx}$

as a solution because all derivatives are alike

except for coefficient. Then;

$$e^{mx}(am^2 + bm + c) = 0$$

But $e^{mx} \neq 0$

So;

$$(am^2 + bm + c) = 0$$

And is called the characteristics or auxiliary equation of Homog. Or Nonhomog. Eq.
It is obtained as:

$$am^2 + bm + c = 0 \qquad or \qquad aD^2 + bD + c = 0$$

The two roots:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions:

$$y_1 = e^{m_1 x} \qquad and \qquad y_2 = e^{m_2 x}$$

Then:

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

By Theorem 2, this is a complete solution of Homog. equation, if

$$W(y_1, y_2) \neq 0$$

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= e^{m_1 x}(m_2 e^{m_2 x}) - e^{m_2 x}(m_1 e^{m_1 x})$$

$$= (m_2 - m_1) e^{(m_1 + m_2)x} \neq 0$$

Unless $m_1 = m_2$

Example 1:

$$y'' + 7y' + 12y = 0$$

Solution :

$$(D^2 + 7D + 12)y = 0$$

$$m^2 + 7m + 12 = 0$$

$$(m+4)(m+3) = 0$$

$$m_1 = -4 \quad and \quad m_2 = -3$$

$$\therefore y = c_1 e^{-4x} + c_2 e^{-3x}$$

$Example \quad 2:$

$$y'' + 2y' + 5y = 0$$

$Solution:$

$$(D^2 + 2D + 5)y = 0$$

$$m^2 + 2m + 5 = 0$$

$$m^2 + 2m + 1 = -5 + 1$$

$$(m+1)^2 = -4$$

$$m + 1 = \pm 2\sqrt{-1} = \pm 2i$$

$$m_1 = -1 + 2i \quad and \quad m_2 = -1 - 2i$$

$$m_1 = -1 + 2i \quad and \quad m_2 = -1 - 2i$$

$$y = c_1 e^{(-1+2i)x} + c_2 e^{(-1-2i)x}$$

$$y = c_1 e^{-x} e^{2ix} + c_2 e^{-x} e^{-2ix}$$

$$y = e^{-x}(c_1 e^{2ix} + c_2 e^{-2ix})$$

$$= e^{-x}[c_1(\cos 2x + i\sin 2x) + c_2(\cos 2x - i\sin 2x)]$$

$$= e^{-x}[(c_1 + c_2)\cos 2x + i(c_1 - c_2)\sin 2x]$$

$$y = e^{-x}[A\cos 2x + B\sin 2x]$$

$$Euler's\ \ formula$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$Example \;\; 3:$$

$$y'' + 6y' + 9y = 0$$

$$Solution:$$

$$(D^2 + 6D + 9)y = 0$$

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

$$m_1 = m_2 = -3$$

$$\therefore y = c_1 e^{-3x} + c_2 x e^{-3x}$$

Example 4 :

$$y'' - 4y' + 4y = 0$$

$y = 3,\ x = 0$ *and* $y' = 4,\ x = 0$

Solution :

$$(D^2 - 4D + 4)y = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m_1 = m_2 = 2$$

$$\therefore y = c_1 e^{2x} + c_2\, x e^{2x}$$

$$y = c_1 e^{2x} + c_2\, x e^{2x}$$

$$When \quad y = 3, \quad x = 0$$

$$3 = c_1 + 0 \Rightarrow \; c_1 = 3$$

$$When \quad y' = 4, \;\; x = 0$$

$$y' = 2c_1 e^{2x} + c_2 (2xe^{2x} + e^{2x})$$

$$4 = 2(3)e^{0} + c_2 (0 + e^{0})$$

$$c_2 = -2$$

$$\therefore \; y = 3e^{2x} - 2xe^{2x}$$

$Example\ 5:$

$$4y'' + 16y' + 17y = 0$$

$$y = 1,\ t = 0 \qquad and \qquad y = 0,\ x = \pi$$

$Solution:$

$$(4D^2 + 16D + 17)y = 0$$

$$4m^2 + 16m + 17 = 0$$

$$m^2 + 4m = -\frac{17}{4}$$

$$m^2 + 4m + 4 = -\frac{17}{4} + 4$$

$$(m+2)^2 = -\frac{1}{4}$$

$$m + 2 = \pm\frac{1}{2}i$$

$$m_1 = -2 + \frac{1}{2}i \qquad and \qquad m_2 = -2 - \frac{1}{2}i$$

$$m_1 = -2 + \frac{1}{2}i \quad and \quad m_2 = -2 - \frac{1}{2}i$$

$$y = c_1 e^{(-2+\frac{1}{2}i)t} + c_2 e^{(-2-\frac{1}{2}i)t}$$

$$y = c_1 e^{-2t} e^{\frac{1}{2}it} + c_2 e^{-2t} e^{-\frac{1}{2}it}$$

$$y = e^{-2t}(A\cos\frac{1}{2}t + B\sin\frac{1}{2}t)$$

$$y = 1,\ t = 0 \Rightarrow A = 1$$

$$y = 0,\ t = \pi \Rightarrow B = 0$$

$$\therefore y = e^{-2t}\cos\frac{1}{2}t$$

The complete process for solving the homogeneous equation in all possible cases as following:

$$D.E.\quad ay'' + by' + cy = 0 \;or\; (aD^2 + bD + c)y = 0$$

Characteristic equation:

$$am^2 + bm + c = 0 \quad or \quad aD^2 + bD + c = 0$$

Nature of the roots of characteristic equation	Condition on the coefficients of the characteristic equation	Complete solution of the differential equation
Real and unequal $m_1 \neq m_2$	$b^2 - 4ac > 0$	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
Real and equal $m_1 = m_2$	$b^2 - 4ac = 0$	$y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$
Conjugate complex $m_1 = p + iq$, $m_2 = p - iq$	$b^2 - 4ac < 0$	$y = e^{px}(A\cos qc + B\sin qx)$

$$The \ \ non\text{hom}ogeneous \ \ equations$$

$$ay'' + by' + cy = f(x)$$

$$y = complementary \ \ \ function + particular \ \ \text{int}egral$$

$$Theorem\, 4:$$

$$If \ \ Y_1 \, is \ \ a \ \ solutions \ \ of \ \ [ay'' + by' + cy = R_1(x)]$$

$$and \ Y_2 \, is \ \ a \ \ solutions \ \ of \ \ [ay'' + by' + cy = R_2(x)]$$

$$Then \ \ \ Y = Y_1 + Y_2 \ \ is \ \ a \ \ solution \ \ of$$

$$[ay'' + by' + cy = R_1(x) + R_2(x)]$$

$$Method\ \ of\ \ undetermined\ \ coefficients:$$

$$Example: y'' + 4y' + 3y = 5e^{2x}$$

$$Solution: \quad Particular \quad Integral$$

$$Y = Ae^{2x}$$

$$(4Ae^{2x}) + 4(2Ae^{2x}) + 3(Ae^{2x}) = 5e^{2x}$$

$$15A = 5 \quad \Rightarrow \quad A = \frac{1}{3}$$

$$\therefore\ Particular\ \ Integral\ = Y = \frac{1}{3}e^{2x}$$

$$Example: y'' + 4y' + 3y = 5\sin 2x$$

$$Solution: \quad Particular \quad Integral$$

$$Y = A\sin 2x + B\cos 2x$$

$$(-4A\sin 2x - 4B\cos 2x) +$$

$$4(2A\cos 2x - 2B\sin 2x)$$

$$+3(A\sin 2x + B\cos 2x) = 5\sin 2x$$

$$(-A-8B)\sin 2x+(8A-B)\cos 2x=5\sin 2x$$

$$-A-8B=5$$

$$8A-B=0$$

Solve the above 2 Eqs. getting ;

$$A=-\frac{1}{13} \quad and \quad B=-\frac{8}{13}$$

$$\therefore Y=-\frac{1}{13}\sin 2x-\frac{8}{13}\cos 2x$$

$$Example: y'' + 3y' + 2y = 10e^{3x} + 4x^2$$

$$Solution:$$

$$Y_1 = Ae^{3x}$$

$$Y_2 = Bx^2 + Cx + D$$

$$(9Ae^{3x}) + 3(3Ae^{3x}) + 2(Ae^{3x}) = 10e^{3x}$$

$$20Ae^{3x} = 10e^{3x}$$

$$A = \frac{1}{2}$$

$$2B + 3(2Bx + C) + 2(Bx^2 + Cx + D) = 4x^2$$

$$2B + 6Bx + 3C + 2Bx^2 + 2Cx + 2D = 4x^2$$

$$2Bx^2 + (6B + 2C)x + (2B + 3C + 2D) = 4x^2$$

$$2B = 4 \quad \Rightarrow B = 2$$

$$6B + 2C = 0 \Rightarrow C = -6$$

$$2B + 3C + 2D = 0 \Rightarrow D = 7$$

$$Y = Y_1 + Y_2$$

$$Y = \frac{e^{3x}}{2} + 2x^2 - 6x + 7$$

$$\therefore complete \quad solution$$

$$y = c_1 e^{-2x} + c_2 e^{-x} + Y$$

$$\therefore y = c_1 e^{-2x} + c_2 e^{-x} + \frac{e^{3x}}{2} + 2x^2 - 6x + 7$$

$$Example: y'' + 5y' + 6y = 3e^{-2x} + e^{3x}$$

$$Solution:$$

$$m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$m_1 = -2 \quad and \quad m_2 = -3$$

$$y = c_1 e^{-2x} + c_2 e^{-3x} + Y$$

$$Y = Y_1 + Y_2$$

$$Y_1 = Axe^{-2x}$$

$$Y_2 = Be^{3x}$$

$$Y = Axe^{-2x} + Be^{3x}$$

$$[4Axe^{-2x} - 2Ae^{-2x} - 2Ae^{-2x}] +$$

$$5[-2Axe^{-2x} + Ae^{-2x}] + 6[Axe^{-2x}] = 3e^{-2x}$$

$$4Axe^{-2x} - 4Ae^{-2x} - 10Axe^{-2x} + 5Ae^{-2x} + 6Axe^{-2x} = 3e^{-2x}$$

$$Ae^{-2x} = 3e^{-2x}$$

$$A = 3$$

$$[9Be^{3x}]+5[3Be^{3x}]+6[Be^{3x}]=e^{3x}$$

$$30Be^{3x}=e^{3x}$$

$$B=\frac{1}{30}$$

$$Y=3xe^{-2x}+\frac{1}{30}e^{3x}$$

$$\therefore y=c_1e^{-2x}+c_2e^{-3x}+3xe^{-2x}+\frac{1}{30}e^{3x}$$

$$Example: y'' - 2y' + y = xe^x - e^x$$

$$Solution: \ \ Homog.$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m_1 = \ m_2 = 1$$

$$y = c_1 e^x + c_2 \, x e^x + Y$$

$$Y = Y_1 + Y_2$$

$$Y = A_0 x^3 e^x + A_1 x^2 e^x$$

$$\therefore Y = (A_0 x^3 + A_1 x^2) e^x$$

$$[A_0x^3+6(A_0+A_1)x^2+(6A_0+4A_1)x+2A_1]e^x$$

$$-2[A_0x^3+(3A_0+A_1)x^2+2A_1x]e^x$$

$$+[A_0x^3+A_1x^2]e^x=xe^x-e^x$$

$$6A_0xe^x+2A_1e^x=xe^x-e^x$$

$$6A_0=1 \quad \Rightarrow \quad A_0=\frac{1}{6}$$

$$2A_1=-1 \Rightarrow \quad A_1=-\frac{1}{2}$$

$$Y=(\frac{1}{6}x^3-\frac{1}{2}x^2)e^x$$

$$\therefore y=c_1e^x+c_2xe^x-\frac{1}{2}x^2e^x+\frac{1}{6}x^3e^x$$

Particular integrals by the method of variation of parametric:

$$ay'' + by' + cy = f(x) \qquad (1)$$

$$NonHomo. \quad y'' + P(x)y' + Q(x)y = R(x) \qquad (2)$$

$$Homo.\ Eq. \quad y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

As we do in construction the complementary function, we attempt to find two functions of x, say u_1 and u_2 such that;

$$Y = u_1 y_1 + u_2 y_2$$

Will be a solution of nonhomogeneous equation (2)

Having two unknown functions, we required two equations for their determination:

Eq. (1) by substitution Y into given D.E. (2)

Eq. (2) remains at our disposal.

From $$Y = u_1 y_1 + u_2 y_2$$

By differentiation ; we have:

$$Y' = (u_1 y_1' + y_1 u_1') + (u_2 y_2' + y_2 u_2')$$

Put;

$$u_1' y_1 + u_2' y_2 = 0 \qquad (4)$$

Now, $$Y' = u_1 y_1' + u_2 y_2'$$

And; $$Y'' = (u_1 y_1'' + y_1' u_1') + (u_2 y_2'' + y_2' u_2')$$

$Now,\ Substitution\ Y, Y'\ and\ Y''\ \text{into}\ Eq.\ (2)$

$We\ \ obtain;$

$$(u_1 y_1'' + y_1' u_1' + u_2 y_2'' + y_2' u_2') + P(x)(u_1 y_1' + u_2 y_2')$$
$$+ Q(x)(u_1 y_1 + u_2 y_2) = R(x)$$

$Or;$

$$u_1 (y_1'' + P(x) y_1' + Q(x) y_1) + u_2 (y_2'' + P(x) y_2' + Q(x) y_2)$$
$$+ (u_1' y_1' + u_2' y_2') = R(x)$$

$$\therefore \quad y_1' u_1' + y_2' u_2' = R(x) \qquad (5)$$

By solving Eqs. (4) *and* (5) *for* u_1' *and* u_2', *we obtain;*

$$\left.\begin{aligned} u_1' &= -\frac{y_2}{y_1 y_2' - y_2 y_1'} R(x) \\ u_2' &= \frac{y_1}{y_1 y_2' - y_2 y_1'} R(x) \end{aligned}\right\} \qquad (6)$$

$$Example:\ y'' - 4y = 3x$$

$$Solution:\ \ Homog.\ \ m^2 - 4 = 0$$

$$m = \pm 2$$

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

$$y = u_1(x)e^{2x} + u_2(x)e^{-2x} \qquad\qquad u = f(x)$$

$$y' = 2u_1(x)e^{2x} + u_1'(x)e^{2x} - 2u_2(x)e^{-2x} + u_2'(x)e^{-2x}$$

$$y' = 2u_1(x)e^{2x} - 2u_2(x)e^{-2x} + u_1'(x)e^{2x} + u_2'(x)e^{-2x}$$

$$u_1'(x)e^{2x} + u_2'(x)e^{-2x} = 0 \qquad (1)$$

$$y' = 2u_1 e^{2x} - 2u_2 e^{-2x}$$

$$y'' = 4u_1 e^{2x} + 4u_2 e^{-2x} + 2u_1' e^{2x} - 2u_2' e^{-2x}$$

$Subst.\quad y \, and \;\; y'' \;\; \text{int}\, o \quad D.E.$

$$4u_1 e^{2x} + 4u_2 e^{-2x} + 2u_1' e^{2x} - 2u_2' e^{-2x} - 4(u_1 e^{2x} + u_2 e^{-2x}) = 3x$$

$$2u_1' e^{2x} - 2u_2' e^{-2x} = 3x \qquad (2)$$

By Solving Eqs. (1) and (2) getting:

$$u_1' = \frac{3}{4} x e^{-2x} \qquad and \qquad u_2' = -\frac{3}{4} x e^{2x}$$

$$u_1 = -\frac{3}{8} x e^{-2x} - \frac{3}{16} e^{-ex} + c_1 \qquad and$$

$$u_2 = -\frac{3}{8} x e^{2x} + \frac{3}{16} e^{ex} + c_2$$

Subst. u_1 *and* u_2, *get;*

$$y = -\frac{3}{4} x + c_1 e^{2x} + c_2 e^{-2x}$$

$$Other \quad method:$$

$$By \quad formula$$

$$y_1 = e^{2x} \qquad and \qquad y_2 = e^{-2x}$$

$$u_1' = -\frac{y_2}{y_1 y_2' - y_2 y_1'} R(x)$$

$$u_2' = \frac{y_1}{y_1 y_2' - y_2 y_1'} R(x)$$

$$u_1' = -\frac{e^{-2x}}{e^{2x}(-2e^{-2x}) - (e^{-2x}(2e^{2x}))} * 3x$$

$$u_1' = -\frac{3xe^{-2x}}{-4}$$

$$= \frac{3}{4} xe^{-2x}$$

$$\therefore u_1 = -\frac{3}{4} xe^{-2x} - \frac{3}{16} e^{-2x}$$

$$u_2' = \frac{e^{2x}}{e^{2x}(-2e^{-2x}) - (e^{-2x}(2e^{2x}))} * 3x$$

$$u_2' = \frac{3xe^{2x}}{-4}$$

$$= -\frac{3}{4}xe^{2x}$$

$$\therefore u_2 = -\frac{3}{8}xe^{2x} + \frac{3}{16}e^{2x}$$

$$Y = u_1 y_1 + u_2 y_2$$

$$= \left(-\frac{3}{8} x e^{-2x} - \frac{3}{16} e^{-2x}\right) e^{2x}$$

$$+ \left(-\frac{3}{8} x e^{2x} + \frac{3}{16} e^{2x}\right) e^{-2x}$$

$$\therefore Y = -\frac{3}{4} x$$

$$y = y_{\text{homo}} + Y$$

$$\therefore y = c_1 e^{2x} + c_2 e^{-2x} - \frac{3}{4} x$$

$$Equations\ \ of\ \ higher\ \ order:$$

$$Example:\ \ y''' + 3y'' + 3y' + y = 0$$

$$Solution:$$

$$m^3 + 3m^3 + 3m + 1 = 0$$

$$(m+1)^3 = 0$$

$$m_1 = m_2 = m_3 = -1$$

$$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$$

$$Example:\ (D^4 + 8D^2 + 16)y = 0$$

$$Solution:$$

$$m^4 + 8m^2 + 16 = 0$$

$$(m^2 + 4)^2 = 0$$

$$m^2 = -4$$

$$m = \pm 2i, \quad \pm 2i$$

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x$$

$$Example: y^{iv} - \lambda^4 y = 0$$

$$Solution:$$

$$m^4 - \lambda^4 = 0$$

$$m^4 = \lambda^4$$

$$m^2 = \pm\lambda^2$$

$$m = \pm\lambda, \ \ \pm\lambda i$$

$$y = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + c_3 \cos\lambda x + c_4 \sin\lambda x$$

$$Example: y''' + 5y'' + 9y' + 5y = 3e^{2x}$$

$$Solution: \ Homog.$$

$$m^3 + 5m^2 + 9m + 5 = 0$$

$$(m+1)(m^2 + 4m + 5) = 0$$

$$m^2 + 4m + 5 = 0$$

$$(m+2)^2 = -1$$

$$m + 2 = \pm i \quad \Rightarrow \quad m = -2 \pm i$$

$$m_1=-1,\quad m_2=-2+i \; and \quad m_3=-2-i$$

$$y_{\text{homo}}=c_1e^{-x}+e^{-2x}(A\cos x+B\sin x)$$

$$y=c_1e^{-x}+e^{-2x}(A\cos x+B\sin x)+Y$$

$$Y=Ae^{2x}$$

$$8Ae^{2x}+5(4Ae^{2x})+9(2Ae^{2x})+5(Ae^{2x})=3e^{2x}$$

$$51Ae^{2x}=3e^{2x} \quad\Rightarrow\quad A=\frac{1}{17} \quad\Rightarrow\quad \therefore Y=\frac{1}{17}e^{2x}$$

$$y=c_1e^{-x}+e^{-2x}(A\cos x+B\sin x)+\frac{e^{2x}}{17}$$

Synthetic division ??????

Example $6y^{iv} + 7y''' - 13y'' - 4y' + 4y = 0$

$$6x^4 + 7x^3 - 13x^2 - 4x + 4 = 0$$

$a: \quad \pm 1, \quad \pm 2, \quad \pm 4$

$b: \quad \pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6$

$$\frac{a}{b} = \pm 1, \pm 2, \pm 4, \ \pm\frac{1}{2}, \ \pm\frac{1}{3}, \ \pm\frac{1}{6}, \ \pm\frac{2}{3}, \ \pm\frac{4}{3}$$

$$+1, \ -2, \ +\frac{1}{2}, \ -\frac{2}{3}$$

1	6	+7	-13	-4	+4
		+6	+13	0	-4
-2	6	13	0	-4	0
		-12	-2	+4	
1/2	6	+1	-2	0	
		+3	+2		
-2/3	6	+4	0		
	6	-4			
	6	0			

$$6x^4 + 7x^3 - 13x^2 - 4x + 4 = 0$$

$$6(x-1)(x+2)(x-\frac{1}{2})(x+\frac{2}{3}) = 0$$

$$Example:\ (D^4+8D^2+16)y=-\sin x$$

$$Solution:$$

$$m^4+8m^2+16=0$$

$$(m^2+4)^2=0$$

$$m^2=-4$$

$$m=\pm 2i,\ \ \pm 2i$$

$$y=c_1\cos 2x+c_2\sin 2x+c_3x\cos 2x+c_4x\sin 2x+Y$$

$$Y = A\cos x + B\sin x$$

$$(A\cos x + B\sin x) + 8(-A\cos x - B\sin x) +$$

$$16(A\cos x + B\sin x) = -\sin x$$

$$9A\cos x + 9B\sin x = -\sin x$$

$$A = 0 \qquad and \qquad B = -\frac{1}{9}$$

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 x\cos 2x + c_4 x\sin 2x - \frac{1}{9}\sin x$$

$$Operator \quad Method:$$

$$\frac{dy}{dx} = Dy, \quad \frac{d^2y}{dx^2} = D^2y, \quad \frac{d^ny}{dx^n} = D^ny$$

$$Ex: \ find \ \ (D^2 + 2D + 1)y \qquad If \qquad y = x^2 + x$$

$$D^2y + 2Dy + y = 2 + 2(2x + 1) + x^2 + x$$

$$= 4 + 5x + x^2$$

$$Define \quad D^{-1}$$

$$Let \quad D^{-1}f(x) = u$$

$$DD^{-1}f(x) = Du$$

$$\therefore Du = f(x)$$

$$\frac{du}{dx} = f(x)$$

$$u = \int f(x)dx$$

$$Theorem \quad f(D) = aD^2 + bD + c$$

$$f(D)e^{mx} = am^2e^{mx} + bme^{mx} + ce^{mx}$$

$$f(D)e^{mx} = e^{mx}(am^2 + bm + c) = e^{mx} f(m)$$

$$(aD^2 + bD + c)y = e^{mx}$$

$$\therefore y = \frac{e^{mx}}{aD^2 + bD + c} \qquad D = m$$

$Ex: Solve;$

$$\frac{d^2y}{dx^2}+y=4e^{2x}$$

$Solution:$ $\quad (D^2+1)y=4e^{2x}$

$Homo.$ $\quad D^2+1=0 \Rightarrow D=\pm i$

$$y_{\text{homo}}=c_1\cos x+c_2\sin x$$

$$y_p=\frac{4e^{2x}}{(2)^2+1}=\frac{4}{5}e^{2x}$$

$$\therefore y=c_1\cos x+c_2\sin x+\frac{4}{5}e^{2x}$$

For;

$$f(x)=\sin Bx \quad or \quad f(x)=\cos Bx$$

$$(aD^2+bD+c)y=f(x)$$

$$y_p=\frac{f(x)\begin{bmatrix}\cos Bx\\ \sin Bx\end{bmatrix}}{aD^2+bD+c}$$

$$Put: \quad D^2=-B^2$$

$Ex: Solve;$

$$(D^2-1)y=\sin 2x$$

$Solution:$

$Homo. \quad D^2-1=0 \Rightarrow D=\pm 1$

$$y_{\text{homo}}=c_1e^{x}+c_2e^{-x}$$

$$y_p=\frac{\sin 2x}{D^2-1}=\frac{\sin 2x}{-2^2-1}=-\frac{1}{5}\sin 2x$$

$$\therefore\ y=c_1e^{x}+c_2e^{-2x}-\frac{1}{5}\sin 2x$$

$$Euler - Equation:$$

$$Try\ general\ form;$$

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=f(x)........(1)\quad 2^{nd}\ order$$

$$Solution:$$

$$Let\ \ t=\ln x$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{1}{x}\times\frac{dy}{dt}\ \Rightarrow\ \frac{dy}{dt}=x\ \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dt^2}\frac{1}{x} + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$Subst.\ \text{int}\,o\ \ Eq.(1)$$

$$a\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + b\frac{dy}{dt} + cy = g(t)$$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = g(t)$$

$$2^{nd}\ order\ \ with\ \ coefficien\,ts$$

$$Example : Solve$$

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = x$$

$$Solution :$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{x} \quad \frac{dy}{dt}$$

$$\frac{dy}{dt} = x \ \frac{dy}{dx}$$

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$$

$$\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+3\frac{dy}{dt}+y=e^t$$

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=e^t$$

$$Homog. \quad m^2+2m+1=0$$

$$y_h=c_1e^{-t}+c_2te^{-t}$$

$$y_p=\frac{e^t}{(1)^2+2(1)+1}=\frac{1}{4}e^t$$

$$y=c_1e^{-t}+c_2te^{-t}+\frac{e^t}{4}$$

$$y=\frac{c_1}{x}+c_2\frac{\ln x}{x}+\frac{x}{4}$$

Applications:

Buckling of columns;

A slender column fails under compressive force when reach a certain value this is called "buckling load"

From strength of materials

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

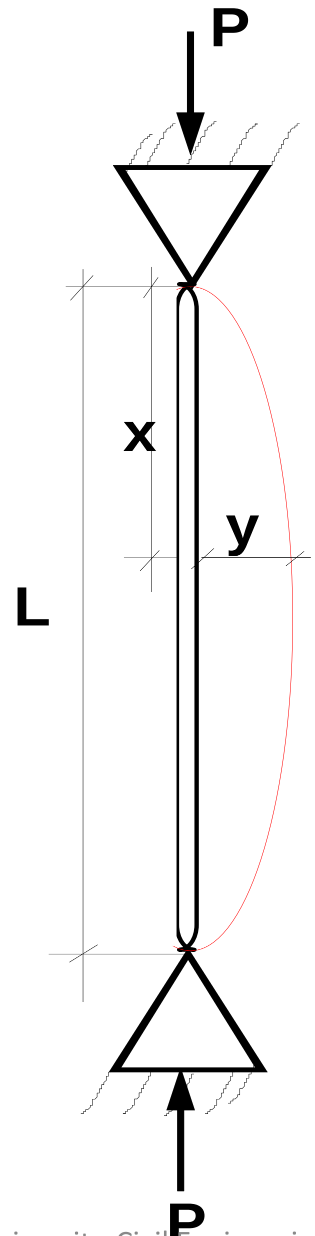
$$1-\text{Buckling of columns hinged at both ends;}$$

$$\frac{d^2y}{dx^2}=-\frac{M}{EI}$$

$$\text{At distance } x,\ M=Py$$

$$\frac{d^2y}{dx^2}=-\frac{Py}{EI}$$

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=0$$


P
x
y
L
P

$$\frac{P}{EI}=k^2$$

$$\frac{d^2y}{dx^2}+k^2y=0$$

$$m^2+k^2=0 \Rightarrow m=\pm ki$$

$$y=c_1\cos kx+c_2\sin kx$$

$$B.Cs.$$

$$y=0 \quad for \quad x=0 \quad and \quad x=L$$

$$y = 0 \quad for \quad x = 0$$

$$0 = c_1 \cos 0 + 0 \Rightarrow c_1 = 0$$

$$y = c_2 \sin kx$$

$$y = 0 \quad for \quad x = L$$

$$0 = c_2 \sin kL + 0 \Rightarrow c_2 \neq 0$$

$$\therefore \sin kL = 0 \Rightarrow kL = 0, \ \pi, \ 2\pi, \ldots$$

$$kL = \pi \quad \Rightarrow k^2 L^2 = \pi^2 \Rightarrow k^2 = \frac{\pi^2}{L^2}$$

$$kL = \pi \;\Rightarrow k^2L^2 = \pi^2 \Rightarrow k^2 = \frac{\pi^2}{L^2}$$

$$\frac{P}{EI} = \frac{\pi^2}{L^2}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

P_{cr} : *buckling load*

L : *length of column*

E : modulus of elasticity

I : *moment of inertia*

$2-$ *Buckling of columns fixed at both ends;*

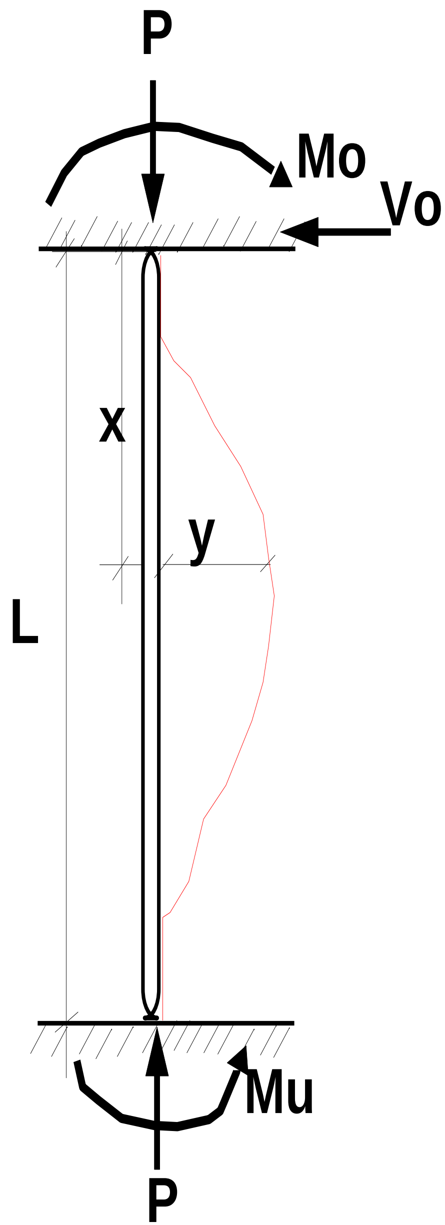
$$V_o = 0$$

$$M = Py - M_o$$

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

$$\frac{d^2y}{dx^2} = -\frac{(Py - M_o)}{EI}$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M_o}{EI}$$


P
Mo
Vo
x
y
L
Mu
P

$$\frac{P}{EI} = k^2$$

$$\frac{d^2y}{dx^2} + k^2 y = \frac{M_o}{EI}$$

$$m^2 + k^2 = 0 \Rightarrow m = \pm ki$$

$$y_h = c_1 \cos kx + c_2 \sin kx$$

$$\frac{d^2y}{dx^2} + k^2 y = \frac{M_o}{EI}$$

$$(D^2 + k^2)y = \frac{M_o}{EI}$$

$$(D^2+k^2)y=\frac{M_o}{EI}$$

$$y_p=\frac{M_o e^0}{EI(D^2+k^2)}$$

$$y_p=\frac{M_o}{EI(0+k^2)}=\frac{M_o}{EIk^2}=\frac{M_o}{EI(\frac{P}{EI})}$$

$$\therefore y_p=\frac{M_o}{P}$$

$$y=c_1\cos kx+c_2\sin kx+\frac{M_o}{P}$$

$B.Cs.$

$$y=0 \quad for \quad x=0 \quad and \quad x=L$$

$$\frac{dy}{dx}=0 \quad for \quad x=0 \quad and \quad x=L$$

$$\frac{dy}{dx} = -c_1 k \sin kx + c_2 k \cos kx$$

$$\frac{dy}{dx} = 0 \quad for \quad x = 0$$

$$0 = 0 + c_2 k \cos kx \implies c_2 = 0$$

$$y = c_1 \cos kx + \frac{M_o}{P}$$

$$y = 0 \quad for \quad x = 0$$

$$0 = c_1 (1) + \frac{M_o}{P} \implies c_1 = -\frac{M_o}{P}$$

$$y = -\frac{M_o}{P} \cos kx + \frac{M_o}{P}$$

$$\therefore y = \frac{M_o}{P} (1 - \cos kx)$$

$$y = 0 \quad for \quad x = L$$

$$0 = \frac{M_o}{P}(1 - \cos kL)$$

$$either \quad \frac{M_o}{P} = 0 \quad or \quad 1 - \cos kL = 0$$

$$\therefore \cos kL = 1 \Rightarrow kL = 0,\ 2\pi,\ 4\pi, \ldots\ldots$$

$$kL = 2\pi$$

$$k^2 L^2 = 4\pi^2 \Rightarrow \frac{P}{EI} L^2 = 4\pi^2$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{(\frac{1}{2}L)^2}$$

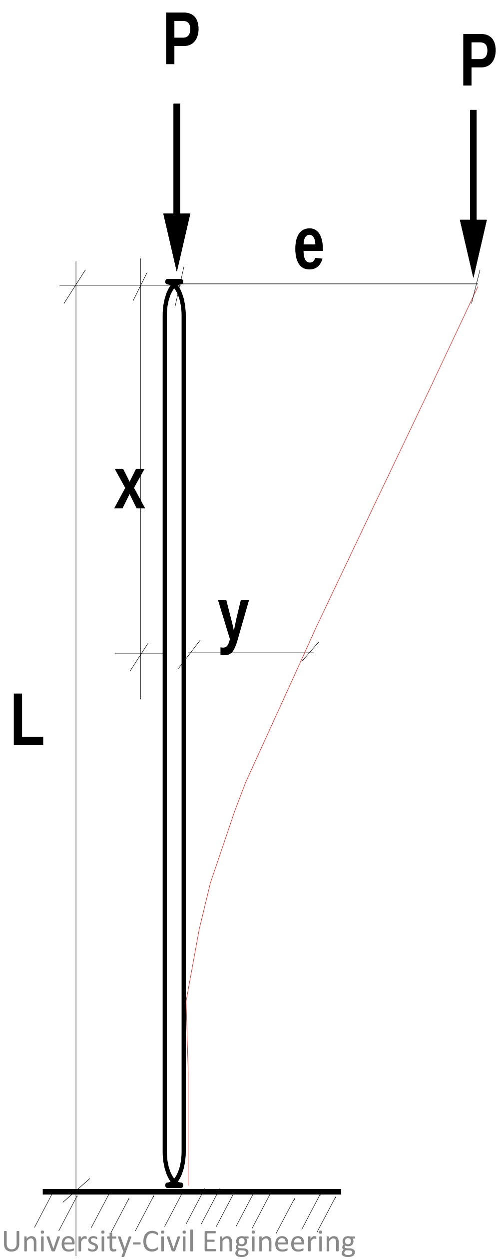
$3-$ *Buckling of columns fixed at the bottom, free at the top;*

$$M=-P(e-y)$$

$$\frac{d^2y}{dx^2}=-\frac{M}{EI}$$

$$\frac{d^2y}{dx^2}=\frac{P(e-y)}{EI}$$

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=\frac{Pe}{EI}$$


P
P
e
x
y
L

$$Let\ \frac{P}{EI} = k^2$$

$$\frac{d^2y}{dx^2} + k^2 y = k^2 e$$

$$m^2 + k^2 = 0 \Rightarrow m = \pm ki$$

$$y_h = c_1 \cos kx + c_2 \sin kx$$

$$y_p = A\ , \quad y_p'' = 0$$

$$0 + k^2 A = k^2 e \Rightarrow A = e$$

$$\therefore y_p = e$$

$$y = c_1 \cos kx + c_2 \sin kx + e$$

$B.Cs.$

$$y = 0 \quad for \quad x = L$$

$$y = e \quad for \quad x = 0$$

$$\frac{dy}{dx} = 0 \quad for \quad x = L$$

$$y = c_1 \cos kx + c_2 \sin kx + e$$

$$y = e \quad for \quad x = 0$$

$$e = c_1(1) + 0 + e \quad \Rightarrow c_1 = 0$$

$$y = c_2 \sin kx + e$$

$$\frac{dy}{dx} = c_2 k \cos kx$$

$$\frac{dy}{dx} = 0 \quad for \quad x = L$$

$$0 = c_2 k \cos kL$$

$$c_2 k \neq 0$$

$$\therefore \cos kL = 0 \quad \Rightarrow kL = \frac{\pi}{2}$$

$$k^2 L^2 = \frac{1}{4}\pi^2$$

$$\frac{P}{EI} L^2 = \frac{1}{4}\pi^2$$

$$P_{cr} = \frac{EI\pi^2}{(2L)^2}$$