

Simul tan eous Linear Differential Equations

$$x = x(t)$$

$$y = y(t)$$

$$a Dx + b Dy = F_1(t) \dots \dots \dots (1)$$

$$c Dx + d Dy = F_2(t) \dots \dots \dots (2)$$

$$\begin{bmatrix} aD & bD \\ cD & dD \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} F_1(t) & bD \\ F_2(t) & dD \end{vmatrix}}{\begin{vmatrix} aD & bD \\ cD & dD \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} aD & F_1(t) \\ cD & F_2(t) \end{vmatrix}}{\begin{vmatrix} aD & bD \\ cD & dD \end{vmatrix}}$$

Ex.: Solve

$$2\frac{dx}{dt} + x + \frac{dy}{dt} = t \dots\dots\dots(1)$$

$$3\frac{dx}{dt} + 2\frac{dy}{dt} - y = 0 \dots\dots\dots(2)$$

Solution

$$2Dx + x + Dy = t$$

$$3Dx + 2Dy - y = 0$$

$$(2D + 1)x + Dy = t \quad \dots\dots\dots(1)$$

$$3Dx + (2D - 1)y = 0 \quad \dots\dots\dots(2)$$

$$\begin{bmatrix} (2D + 1) & D \\ 3D & (2D - 1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} t & D \\ 0 & (2D-1) \end{vmatrix}}{\begin{vmatrix} (2D+1) & D \\ 3D & (2D-1) \end{vmatrix}}$$

$$x = \frac{(2D-1)t - 0}{(2D+1)(2D-1) - 3D(D)}$$

$$x = \frac{2Dt - t}{4D^2 - 1 - 3D^2} = \frac{2 - t}{D^2 - 1}$$

$$(D^2 - 1)x = 2 - t$$

$$D^2 - 1 = 0 \Rightarrow D = \pm 1$$

$$x_{\text{homo.}} = c_1 e^t + c_2 e^{-t}$$

$$x_p = A + Bt$$

$$x' = B$$

$$x'' = 0$$

$$0 - (A + Bt) = 2 - t$$

$$-A = 2 \Rightarrow A = -2$$

$$-Bt = -t \Rightarrow B = 1$$

$$\therefore x = c_1 e^t + c_2 e^{-t} - 2 + t$$

$$y = \frac{\begin{vmatrix} (2D+1) & t \\ 3D & 0 \end{vmatrix}}{\begin{vmatrix} (2D+1) & D \\ 3D & (2D-1) \end{vmatrix}} = \frac{0 - 3Dt}{D^2 - 1}$$

$$(D^2 - 1)y = -3$$

$$D^2 - 1 = 0 \Rightarrow D = \pm 1$$

$$y_{\text{homo.}} = c_3 e^t + c_4 e^{-t}$$

$$y_p = A_1$$

$$y' = 0$$

$$y'' = 0$$

$$0 - A_1 = -3 \Rightarrow A_1 = 3$$

$$\therefore y = c_3 e^t + c_4 e^{-t} + 3$$

$$\frac{dx}{dt} = Dx = c_1 e^t - c_2 e^{-t} + 1$$

$$\frac{dy}{dt} = Dy = c_3 e^t - c_4 e^{-t}$$

Subst. into Eq. (2)

$$3(c_1 e^t - c_2 e^{-t} + 1) + 2(c_3 e^t - c_4 e^{-t}) - (c_3 e^t + c_4 e^{-t} + 3) = 0$$

$$\text{Coeff. of } e^t \Rightarrow 3c_1 + c_3 = 0 \Rightarrow c_3 = -3c_1$$

$$\text{Coeff. of } e^{-t} \Rightarrow -3c_2 - 3c_4 = 0 \Rightarrow c_4 = -c_2$$

$$\therefore x = c_1 e^t + c_2 e^{-t} - 2 + t$$
$$y = -3c_1 e^t - c_2 e^{-t} + 3$$

Ex.: Solve

$$2\frac{dx}{dt} + \frac{dy}{dt} - x = e^t \dots\dots\dots(1)$$

$$3\frac{dx}{dt} + 2\frac{dy}{dt} + y = t \dots\dots\dots(2)$$

Solution :

$$(2D - 1)x + Dy = e^t$$

$$3Dx + (2D + 1)y = t$$

$$\begin{bmatrix} (2D - 1) & D \\ 3D & (2D + 1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^t \\ t \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} e^t & D \\ t & (2D+1) \end{vmatrix}}{\begin{vmatrix} (2D-1) & D \\ 3D & (2D+1) \end{vmatrix}}$$

$$x = \frac{(2D+1)e^t - Dt}{(2D-1)(2D+1) - 3D(D)}$$

$$x = \frac{2De^t + e^t - Dt}{4D^2 - 1 - 3D^2} = \frac{3e^t - 1}{D^2 - 1}$$

$$(D^2 - 1)x = 3e^t - 1 \dots\dots\dots(1)$$

For y:

$$(D^2 - 1)y = (2D - 1)t - 3De^t$$
$$= 2 - t - 3e^t$$

$$(D^2 - 1)y = -3e^t - t + 2 \dots\dots\dots(2)$$

Roots : ± 1

$$x = c_1 e^t + c_2 e^{-t} + X$$

$$y = c_1 e^t + c_2 e^{-t} + Y$$

$$X = Ate^t + B$$

$$Ate^t + 2Ae^t - Ate^t - B = 3e^t - 1$$

$$A = \frac{3}{2} \text{ and } B = 1$$

$$Y = A_o te^t + B_o t + C_o$$

$$A_o te^t + 2A_o e^t - A_o te^t - B_o t - C_o = -3e^t - t + 2$$

$$A_o = -\frac{3}{2}, \quad B_o = 1 \text{ and } C_o = -2$$

$$\therefore x = c_1 e^t + c_2 e^{-t} + \frac{3}{2} t e^t + 1$$

$$y = c_3 e^t + c_4 e^{-t} - \frac{3}{2} t e^t + t - 2$$

To reduce the No. of constants Subst. x, y and their derivatives in the first equation, (or in second)

$$2\frac{dx}{dt} + \frac{dy}{dt} - x = e^t \dots\dots\dots(1)$$

$$2c_1e^t - 2c_2e^{-t} + 3te^t + 3e^t +$$

$$c_3e^t - c_4e^{-t} - \frac{3}{2}te^t - \frac{3}{2}e^t + 1$$

$$- c_1e^t - c_2e^{-t} - \frac{3}{2}te^t - 1 = e^t$$

Coeff. of e^t :

$$2c_1 + c_3 - c_1 + \frac{3}{2} = 1 \text{ or } c_1 + c_3 + \frac{1}{2} = 0$$

$$\text{or } c_3 = -\frac{1}{2} - c_1$$

Coeff. of e^{-t} :

$$-2c_2 + c_4 - c_2 = 0 \text{ or } -c_2 - c_4 = 0$$

$$\text{or } c_4 = -3c_2$$

$$\therefore x = c_1 e^t + c_2 e^{-t} + \frac{3}{2} t e^t + 1$$

$$y = -c_1 e^t - 3c_2 e^{-t} - \frac{3}{2} t e^t - \frac{1}{2} e^t + t - 2$$

Ex.: Solve

$$(2D + 3D - 9)x + (D^2 + 7D - 14) = 4 \dots\dots\dots(1)$$

$$(D + 1)x + (D + 2)y = 0 \dots\dots\dots(2)$$

Solution

$$\begin{vmatrix} 2D^2 + 3D - 9 & D^2 + 7D - 14 \\ D + 1 & D + 2 \end{vmatrix} x = \begin{vmatrix} 4 & D^2 + 7D - 14 \\ -8e^{2t} & D + 2 \end{vmatrix}$$

$$[(2D^3 + 3D^2 - 9D + 4D^2 + 6D - 18) - (D^3 + 7D^2 - 14D + D^2 - 7D - 14)]x = (D + 2)4 - (D^2 + 7D - 14)(-8e^{2t})$$

$$(D^3 - D^2 + 4D - 4)x = 8 + 32e^{2t}$$

$$(D^3 - D^2 + 4D - 4)x = 8 + 32e^{2t}$$

The roots of the characteristics Eq. are $\pm 2i$ and 1

$$x = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + X$$

$$x = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + Ae^{2t} + B$$

$$x = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + 4e^{2t} - 2$$

$$\begin{vmatrix} 2D^2 + 3D - 9 & D^2 + 7D - 14 \\ D + 1 & D + 2 \end{vmatrix} y = \begin{vmatrix} 2D^2 + 3D - 9 & 4 \\ D + 1 & -8e^{2t} \end{vmatrix}$$

$$(D^3 - D^2 + 4D - 4)y = -40e^{2t} - 4$$

$$y = k_1 \cos 2t + k_2 \sin 2t + k_3 e^t + Y$$

$$y = k_1 \cos 2t + k_2 \sin 2t + k_3 e^t - 5e^{2t} + 1$$

We must have 3 constants, used 2nd Eq.

$$(D+1)(c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + 4e^{2t} - 2) + \\ (D+2)(k_1 \cos 2t + k_2 \sin 2t + k_3 e^t - 5e^{2t} + 1) = -8e^{2t}$$

OR:

$$-2c_1 \sin 2t + 2c_2 \cos 2t + c_3 e^t + 8e^{2t} + \\ c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + 4e^{2t} - 2 + \\ -2k_1 \sin 2t + 2k_2 \cos 2t + k_3 e^t - 10e^{2t} + \\ 2k_1 \cos 2t + 2k_2 \sin 2t + 2k_3 e^t - 10e^{2t} + 2 = -8e^{2t}$$

OR:

$$(c_1 + 2c_2 + 2k_1 + 2k_2) \cos 2t +$$
$$(-2c_1 + c_2 - 2k_1 + 2k_2) \sin 2t +$$
$$(2c_3 + 3k_3) e^t - 8e^{2t} = -8e^{2t}$$

$$\therefore c_1 + 2c_2 + 2k_1 + 2k_2 = 0$$

$$-2c_1 + c_2 - 2k_1 + 2k_2 = 0$$

$$2c_3 + 3k_3 = 0$$

Solve:

$$k_1 = \frac{-3c_1 - c_2}{4}, \quad k_2 = \frac{c_1 - 3c_2}{4} \quad \text{and} \quad k_3 = -\frac{2}{3}c_3$$

$$\begin{aligned}\therefore x &= c_1 \cos 2t + c_2 \sin 2t + c_3 e^t + 4e^{2t} - 2 \\ y &= -\frac{1}{4}(3c_1 + c_2) \cos 2t + \frac{1}{4}(c_1 - 3c_2) \sin 2t \\ &\quad - \frac{2}{3}c_3 e^t - 5e^{2t} + 1\end{aligned}$$

Complementary functions and particular integrals for system of equations :

Ex. Solve:

$$\left. \begin{aligned} (D+1)x + (D+2)y + (D+3)z &= -e^{-t} \\ (D+2)x + (D+3)y + (2D+3)z &= e^{-t} \\ (4D+6)x + (5D+4)y + (20D-12)z &= 7e^{-t} \end{aligned} \right] \dots(1)$$

Solution : As in the case of a single equation make system homogeneous by neglected the terms on the right, getting :

$$\left[\begin{array}{l} (D+1)x + (D+2)y + (D+3)z = 0 \\ (D+2)x + (D+3)y + (2D+3)z = 0 \\ (4D+6)x + (5D+4)y + (20D-12)z = 0 \end{array} \right] \dots(2)$$

Let us now attempt to find solution of this system of the form;

$$x = ae^{mt} \quad , \quad y = be^{mt} \quad , \quad z = ce^{mt} \quad \dots\dots\dots(3)$$

Substituting these into the equation (2) and dividing out the common factor e^{mx} , getting ;

$$\left[\begin{array}{l} (m+1)a + (m+2)b + (m+3)c = 0 \\ (m+2)a + (m+3)b + (2m+3)c = 0 \\ (4m+6)a + (5m+4)b + (20m-12)c = 0 \end{array} \right] \dots(4)$$

If nontrivial solution for x , y and z , solutions that do not vanish identically, are to be obtained, it is necessary that a , b and c shall not all be zero.

However, the values $a = b = c = 0$ satisfy the system (4) and in general will be the only solution.

No other solutions are possible unless the determinant in (4) is equal to zero.

$$\begin{vmatrix} m+1 & m+2 & m+3 \\ m+2 & m+3 & 2m+3 \\ 4m+6 & 5m+4 & 20m-12 \end{vmatrix} = 0$$

Therefor : $-(m-1)(m-2)(m-3) = 0$

The roots are : $m_1 = 1$, $m_2 = 2$ and $m_3 = 3$

\therefore

$$x_1 = a_1 e^t, \quad x_2 = a_2 e^{2t}, \quad x_3 = a_3 e^{3t}$$

$$y_1 = b_1 e^t, \quad x_2 = b_2 e^{2t}, \quad x_3 = b_3 e^{3t}$$

$$z_1 = c_1 e^t, \quad x_2 = c_2 e^{2t}, \quad x_3 = c_3 e^{3t}$$

For $m_1 = 1$, from (4)

$$2a_1 + 3b_1 + 4c_1 = 0$$

$$3a_1 + 4b_1 + 5c_1 = 0$$

$$10a_1 + 9b_1 + 8c_1 = 0$$

These Eqs. are nontrivially solvable,

i.e. has a solution other than;

$$a_1 = b_1 = c_1 = 0$$

For all values of k_1 ; $a_1 = -k_1$, $b_1 = 2k_1$ and $c_1 = -k_1$

Therefore; $x_1 = -k_1 e^t$, $y_1 = -2k_1 e^t$ and $z_1 = -k_1 e^t$

Similarly for $m_2 = 2$, from (4)

$$3a_2 + 4b_2 + 5c_2 = 0$$

$$4a_2 + 5b_2 + 7c_2 = 0$$

$$14a_2 + 14b_2 + 28c_2 = 0$$

For all values of k_2 ; $a_2 = 3k_2$, $b_2 = -k_2$ and $c_2 = -k_2$

Therefore; $x_2 = 3k_2 e^{2t}$, $y_2 = -k_2 e^{2t}$ and $z_2 = -k_2 e^{2t}$

For $m_3 = 3$,

$$4a_3 + 5b_3 + 6c_3 = 0$$

$$5a_3 + 6b_3 + 9c_3 = 0$$

$$18a_3 + 19b_3 + 48c_3 = 0$$

From these ;

$$a_3 = 9k_3, b_3 = -6k_3 \text{ and } c_3 = -k_3$$

Therefore;

$$x_3 = 9k_3 e^{3t}, y_3 = -6k_3 e^{3t} \text{ and } z_3 = -k_3 e^{3t}$$

*Since Eqs. of Homogeneous (2), are linear;
Sum. of solutions will be also be solutions, So;*

$$\begin{aligned}x &= x_1 + x_2 + x_3 \\ &= -k_1 e^t + 3k_2 e^{2t} + 9k_3 e^{3t}\end{aligned}$$

$$\begin{aligned}y &= y_1 + y_2 + y_3 \\ &= 2k_1 e^t - k_2 e^{2t} - 6k_3 e^{3t}\end{aligned}$$

$$\begin{aligned}z &= z_1 + z_2 + z_3 \\ &= -k_1 e^t - k_2 e^{2t} - k_3 e^{3t}\end{aligned}$$

This solution is the complementary function.

For complete solution;

$$X = Ae^{-t}, \quad Y = Be^{-t} \quad \text{and} \quad Z = Ce^{-t}$$

Subst. into Eqs. (1), getting ;

$$(B + 2C)e^{-t} = -e^{-t}$$

$$(A + 2B + C)e^{-t} = e^{-t}$$

$$(2A - B - 32C)e^{-t} = 7e^{-t}$$

Hence; $A = 3$, $B = -1$ and $C = 0$

$$X = 3e^{-t}, \quad Y = -e^{-t} \quad \text{and} \quad Z = 0$$

\therefore complete solution;

$$x = -k_1 e^t + 3k_2 e^{2t} + 9k_3 e^{3t} + 3e^{-t}$$

$$y = 2k_1 e^t - k_2 e^{2t} - 6k_3 e^{3t} - e^{-t}$$

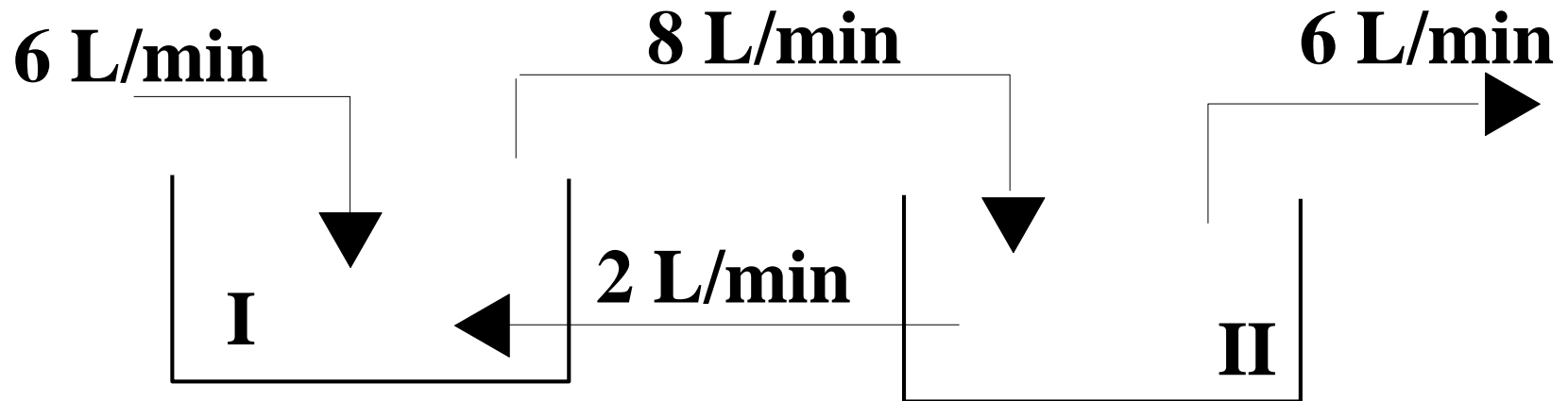
$$z = -k_1 e^t - k_2 e^{2t} - k_3 e^{3t}$$

Applications

Modeling \Rightarrow Solving \Rightarrow Interpolation

Example:

Two tanks are connected as shown below. The first tank contains initially (100L) of brine containing 50N salt. The second tank contains initially (100L) of brine with 20N salt is desolved. Starting at $t=0$, the pumping was applied. If the brine in each tank is kept uniform by stirring. Find the amount of salt in each tank as a function of t .



Solution :

x : Weight of salt in tank I

y : Weight of salt in tank II

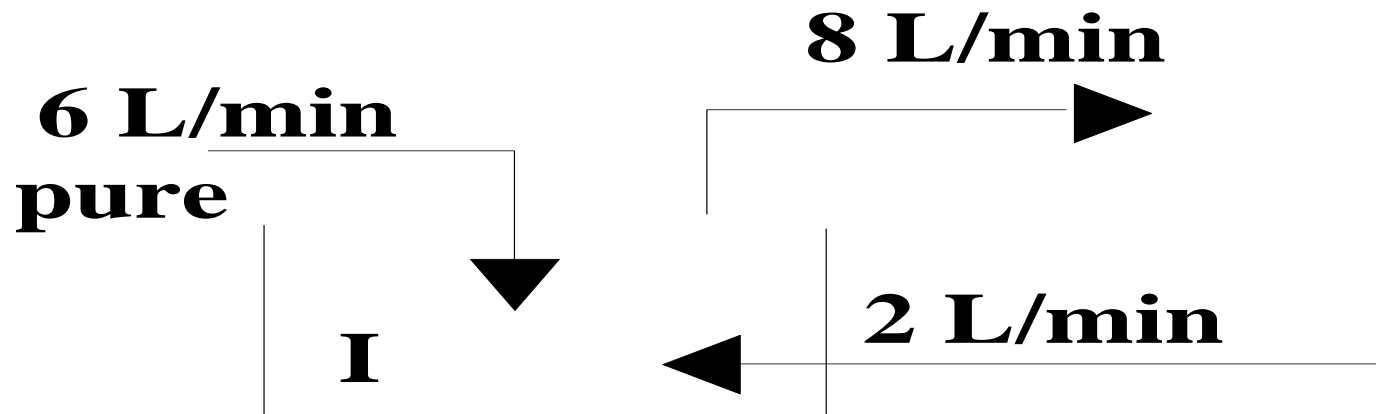
$\frac{x}{100} = \text{concentration of salt in tank I}$

$\frac{y}{100} = \text{concentration of salt in tank II}$

$$\frac{dx}{dt} = 2 \frac{y}{100} - 8 \frac{x}{100}$$

$$\frac{dx}{dt} + 0.08x - 0.02y = 0$$

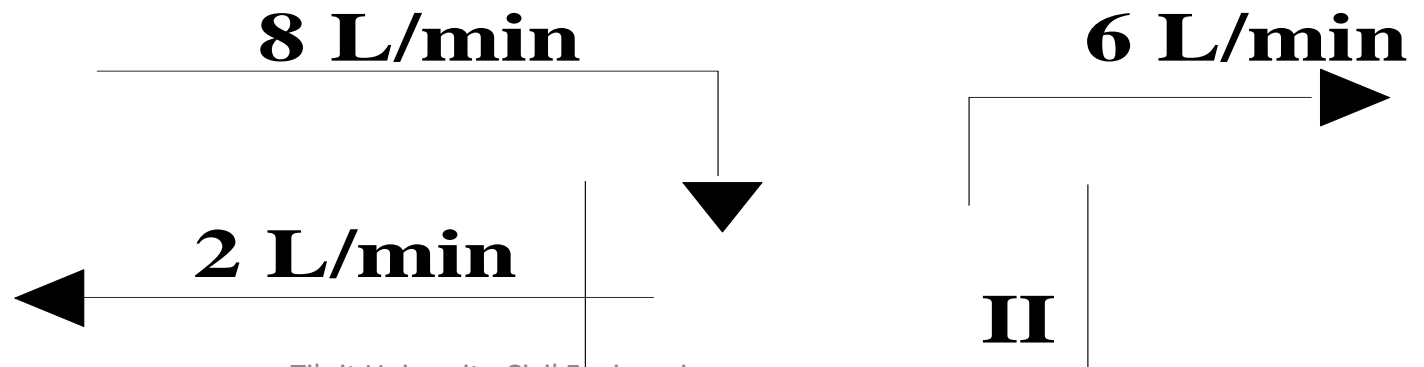
$$(D + 0.08)x - 0.02y = 0 \quad \dots\dots\dots(1)$$



$$\frac{dy}{dt} = 8 \frac{x}{100} - 2 \frac{y}{100} - 6 \frac{y}{100}$$

$$\frac{dy}{dt} = 0.08x - 0.08y$$

$$0.008x - (D + 0.08)y = 0 \quad \text{.....(2)}$$



$$\begin{bmatrix} D + 0.08 & -0.02 \\ 0.08 & -(D + 0.08) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 0 & -0.02 \\ 0 & -(D + 0.08) \end{vmatrix}}{\begin{vmatrix} D + 0.08 & -0.02 \\ 0.08 & -(D + 0.08) \end{vmatrix}}$$

$$[\{(D + 0.08)(-(D + 0.08))\} - (0.08(-0.02))]x = 0$$

$$[-(D^2 + 0.16D + 0.0064) + 0.0016]x = 0$$

$$[-(D^2 + 0.16D + 0.0064) + 0.0016]x = 0$$

$$(-D^2 - 0.16D - 0.0048)x = 0$$

$$(D^2 + 0.16D + 0.0048)x = 0$$

$$(D + 0.12)(D + 0.04)x = 0$$

The roots are; -0.12 and -0.04

$$\therefore x = c_1 e^{-0.12t} + c_2 e^{-0.04t}$$

$$\text{Also; } y = c_3 e^{-0.12t} + c_4 e^{-0.04t}$$

$$x = c_1 e^{-0.12t} + c_2 e^{-0.04t}$$

$$y = c_3 e^{-0.12t} + c_4 e^{-0.04t}$$

Subst. into Eq. (1)

$$(D + 0.08)x - 0.02y = 0$$

$$(-0.12c_1 e^{-0.12t} - 0.04c_2 e^{-0.04t}) +$$

$$0.08(c_1 e^{-0.12t} + c_2 e^{-0.04t}) -$$

$$0.02(c_3 e^{-0.12t} + c_4 e^{-0.04t}) = 0$$

By equating the coefficients :

$$-0.12c_1 + 0.08c_1 - 0.02c_3 = 0$$

$$-0.04c_2 + 0.08c_2 - 0.02c_4 = 0$$

$$\therefore c_3 = \frac{-0.04}{0.02} c_1 = -2c_1$$

$$c_4 = \frac{0.04}{0.02} c_2 = 2c_2$$

$$\therefore x = c_1 e^{-0.12t} + c_2 e^{-0.04t}$$

$$y = -2c_1 e^{-0.12t} + 2c_2 e^{-0.04t}$$

$$x = c_1 e^{-0.12t} + c_2 e^{-0.04t}$$

$$y = -2c_1 e^{-0.12t} + 2c_2 e^{-0.04t}$$

At $t = 0$, $x = 50$ and $y = 20$

$$50 = c_1 + c_2$$

$$20 = -2c_1 + 2c_2$$

$$c_1 = 20 \text{ and } c_2 = 30$$

$$\therefore x = 20e^{-0.12t} + 30e^{-0.04t}$$

$$y = -40e^{-0.12t} + 60e^{-0.04t}$$

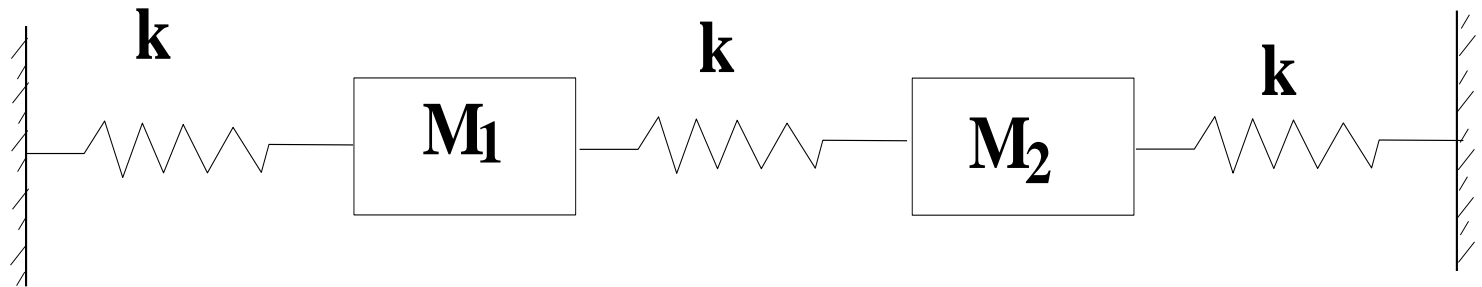
Mechanical vibration

Example:

The system with two degree of freedom beings to move under the following condition;

Initial velocity $\dot{x}_1(0) = \sqrt{3k}$ and $\dot{x}_2(0) = \sqrt{3k}$

Neglected the friction, derive the D.E. covering free vibration of system show below:



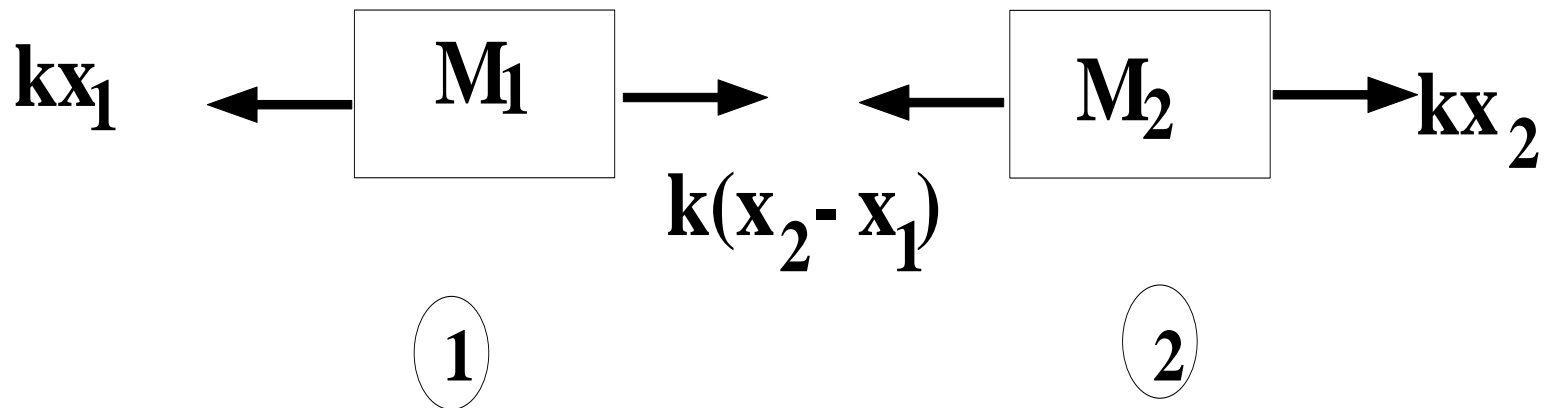
$$M_1 = M_2 = 1$$

$k = \text{spring constant}$

Solution :

*Mass * acceleration = \sum force in the direction of motion*

*Force = spring constant * displacement = $k * x$*



From (1)

$$\text{Mass} * \text{acceleration} = \sum fx$$

$$M_1 \ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$

$$M_1 = 1$$

$$\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$(D^2 + 2k)x_1 - kx_2 = 0 \dots\dots\dots(1)$$

From (2)

$$M_2 \ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

$$M_1 = 1$$

$$-kx_1 + (D^2 + 2k)x_2 = 0 \dots\dots\dots(2)$$

$$\begin{bmatrix} D^2 + 2k & -k \\ -k & D^2 + 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(D^4 + 4kD^2 + 4k^2 - k^2)x_1 = 0$$

$$(m^2 + k)(m^2 + 3k) = 0$$

$$m = \pm i\sqrt{k} \quad \text{and} \quad \pm i\sqrt{3k}$$

$$x_1 = c_1 \cos \sqrt{k} t + c_2 \sin \sqrt{k} t + c_3 \cos \sqrt{3k} t + c_4 \sin \sqrt{3k} t$$

$$x_2 = c_5 \cos \sqrt{k} t + c_6 \sin \sqrt{k} t + c_7 \cos \sqrt{3k} t + c_8 \sin \sqrt{3k} t$$

$$x_1 = c_1 \cos \sqrt{k} t + c_2 \sin \sqrt{k} t + c_3 \cos \sqrt{3k} t + c_4 \cos \sqrt{3k} t$$

$$x_2 = c_5 \cos \sqrt{k} t + c_6 \sin \sqrt{k} t + c_7 \cos \sqrt{3k} t + c_8 \cos \sqrt{3k} t$$

Subst. $x_1(t)$ and $x_2(t)$ into Eq.(1)

$$c_5 = c_1, \quad c_6 = c_2, \quad c_7 = -c_3 \quad \text{and} \quad c_8 = -c_4$$

$$\therefore x_1(t) = c_1 \cos \sqrt{k} t + c_2 \sin \sqrt{k} t + c_3 \cos \sqrt{3k} t + c_4 \cos \sqrt{3k} t$$

$$x_2(t) = c_1 \cos \sqrt{k} t + c_2 \sin \sqrt{k} t - c_3 \cos \sqrt{3k} t - c_4 \cos \sqrt{3k} t$$