

Numerical Solutions of Ordinary D.E.

If $\frac{dy}{dx} = f(x, y)$ can't be solved by direct

integration. The numerical solution techniques are used. To describe various numerical methods for the solution of ordinary D.E., we consider the general first order D.E.

$$\frac{dy}{dx} = f(x, y), \text{ with initial condition } y(x_o) = y_o$$

and illustrate the theory with respect to this equation. The methods so developed can, in general, be applied to the solution of systems of first order equation and will yield the solution in of the two forms;

1– A series for y in terms of powers of x , from which the value of y can be obtained by direct substitution.

2– A set of tabulated values of x and y .

Euler's Method :

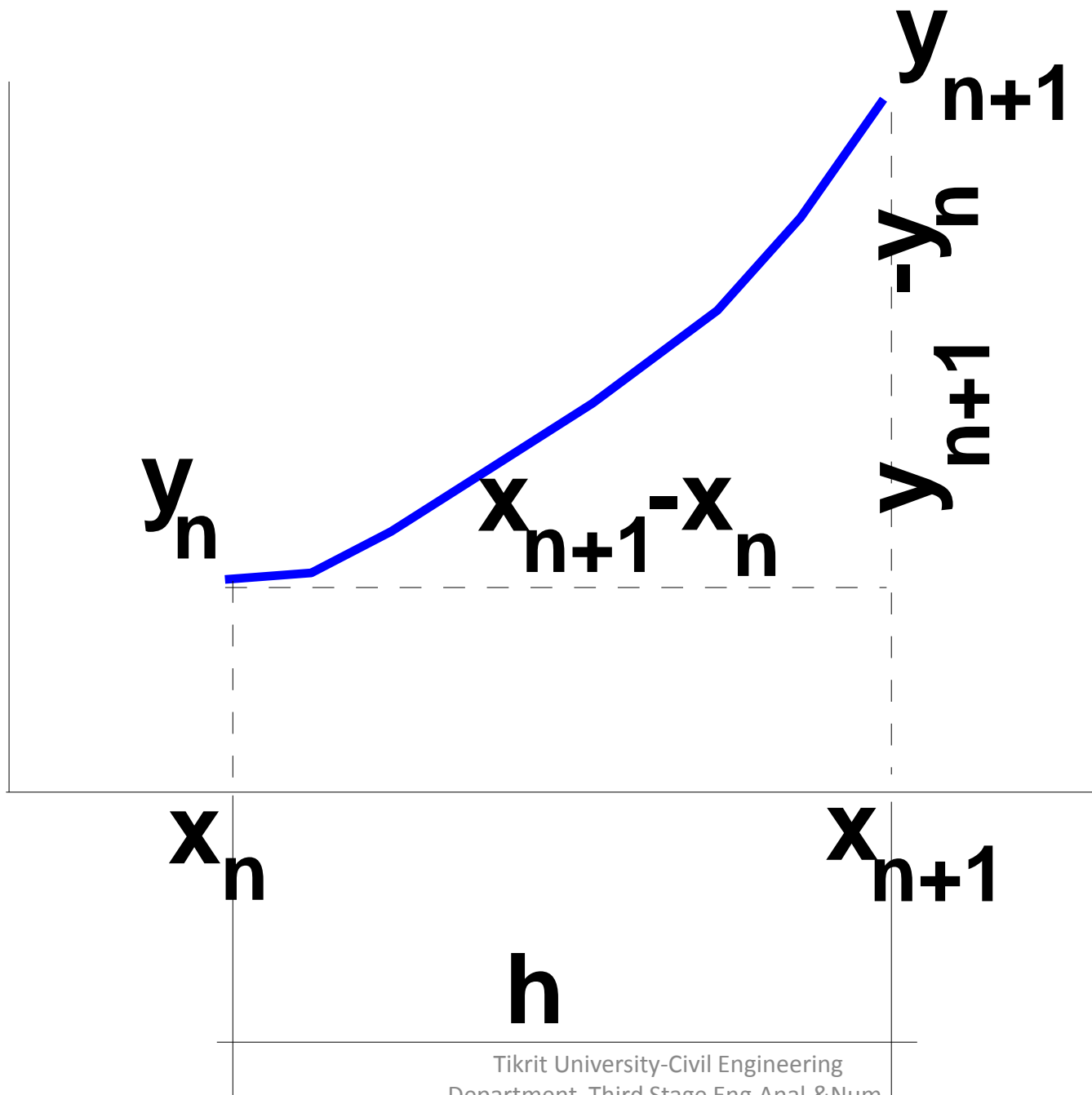
Convert $\frac{dy}{dx}$ to $\frac{\Delta y}{\Delta x}$

$$\frac{dy}{dx} = f(x, y) = f(x_n, y_n)$$

$$f(x_n, y_n) = \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = \frac{y_{n+1} - y_n}{h}$$

$$y_{n+1} - y_n = h * f(x_n, y_n)$$

$$\therefore y_{n+1} = y_n + h f(x_n, y_n)$$



Ex : Consider D.E. $\frac{dy}{dx} = x + y$ with condition

$y(0) = 1$. Find y when $x = 2$ (let $h = 0.5$)

Solution

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_1 = y_o + h f(x_o, y_o)$$

$$y_1 = 1 + 0.5(0 + 1) = 1.5$$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_2 = 1.5 + 0.5(0.5 + 1.5) = 2.5$$

$$y_3 = 2.5 + 0.5(1.0 + 2.5) = 4.25$$

$$y_4 = 4.25 + 0.5(1.5 + 4.25) = 7.125$$

$$\therefore y(2) = 7.125$$

Ex : D.E., $\frac{dy}{dx} = -y$ with $y(0) = 1$.

Find y when $x = 0.04$ (let $h = 0.01$)

Solution

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_1 = y(0.01) = 1 + 0.01(-1) = 0.99$$

$$y_2 = y(0.02) = 0.99 + 0.01(-0.99) = 0.9801$$

$$y_3 = y(0.03) = 0.9801 + 0.01(-0.9801) = 0.9703$$

$$y_4 = y(0.04) = 0.9703 + 0.01(-0.9703) = 0.9606$$

$$\therefore y(0.04) = 0.9606$$

Modified Euler's Method : or

Predictor Corrector Method

This is the simplest of an entire series of method called predictor corrector methods.

1–Predictor step :

$\bar{y}_{n+1} = y_n + \Delta x f(x_n, y_n)$ from Euler's method
having obtained \bar{y}_{n+1} , we now calculate $f(x_{n+1}, \bar{y}_{n+1})$
which is the derivative at the new point
 (x_{n+1}, \bar{y}_{n+1}) and average with the previous
derivative $f(x_n, y_n)$ to find the average derivative;

$$\frac{1}{2} [f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})]$$

2–Now, we substitute this average value into the original iteration equation instead of $f(x_n, y_n)$ to get;

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})]$$

Ex : D.E., $\frac{dy}{dx} = y$, $y(0) = 1$. Find y at $x = 1$

Solution

1– Use $h = 1$, $n = 1$

$$\bar{y}_1 = y_o + h f(x_o, y_o) = 1 + 1(1) = 2$$

$$y_1 = y_o + \frac{h}{2} [f(x_o, y_o) + f(x_1 + \bar{y}_1)]$$

$$= 1 + \frac{1}{2} [1 + 2] = 2.5$$

2- Use $h = 0.5$, $n = 2$

$$\bar{y}_1 = y_o + h f(x_o, y_o) = 1 + 0.5(1) = 1.5$$

$$y_1 = y_o + \frac{h}{2} [f(x_o, y_o) + f(x_1 + \bar{y}_1)]$$

$$= 1 + \frac{0.5}{2} [1 + 1.5] = 1.625$$

$$\bar{y}_2 = y_1 + h f(x_1, y_1) = 1.625 + 0.5(1.625) = 2.4375$$

$$y_2 = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2 + \bar{y}_2)]$$

$$= 1.625 + \frac{0.5}{2} [1.625 + 2.4375] = 2.6406$$

Modified Euler's Method :

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})]$$

$$\bar{y}_{n+1} = y_n + h f(x_n, y_n)$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_{n+1}, y_n + k_1)$$

$$\therefore y_{n+1} = y_n + \frac{1}{2} (k_1 + k_2)$$

Ex : D.E., $\frac{dy}{dx} = y$, $y(0) = 1$. Find y at $x = 1$

use $h = 0.5$

Solution

$$x_o = 0, \quad y_o = 1, \quad h = 0.5$$

$$k_1 = 0.5(1) = 0.5$$

$$k_2 = 0.5(1.5) = 0.75$$

$$y_1 = 1 + \frac{1}{2}(0.5 + 0.75) = 1.625$$

$$x_1 = 0.5, \quad y_1 = 1.625, \quad h = 0.5$$

$$k_1 = 0.5(1.625) = 0.8125$$

$$k_2 = 0.5(1.625 + 0.8125) = 1.21875$$

$$y_1 = 1.625 + \frac{1}{2}(0.8125 + 1.21875) = 2.640625$$

Ex : Determine the value of y when $x = 0.1$

given that $y(0) = 1$ and $y' = x^2 + y$

Solution

$$1 - h = 0.05$$

$$\bar{y}_1 = y_o + hf(x_o, y_o) = 1 + 0.05(0^2 + 1) = 1.05$$

$$y_1 = y_o + \frac{h}{2}[f(x_o, y_o) + f(x_1, \bar{y}_1)]$$

$$= 1 + \frac{0.05}{2}[1 + \{(0.05)^2 + 1.05\}] = 1.0513$$

$$\bar{y}_2 = y_1 + hf(x_1, y_1)$$

$$= 1.0513 + 0.05(0.05^2 + 1.0513) = 1.10399$$

$$y_2 = 1.0513 + \frac{0.05}{2}[(0.05)^2 + 1.0513 + (0.1)^2 + 1.10399] = 1.10549$$

$$2- \quad x_o = 0, \quad y_o = 1, \quad h = 0.05$$

$$k_1 = h f(x_n, y_n) = 0.05(x_o^2 + y_o) = 0.05 + (0^2 + 1) = 0.05$$

$$k_2 = h f(x_{n+1}, y_n + k_1) = 0.05[(0 + 0.5)^2 + (1 + 0.05)] = 0.052625$$

$$y_1 = y_o + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0.05 + 0.052625) = 1.0513$$

$$x_1 = 0.05, \quad y_1 = 1.0513, \quad h = 0.05$$

$$k_1 = 0.05[(0.05)^2 + 1.0513] = 0.05269$$

$$k_2 = 0.05[(0.1)^2 + (1.0513 + 0.05269)] = 0.0557$$

$$y_2 = 1.0513 + \frac{1}{2}(0.05269 + 0.0557) = 1.10549$$

Runge – Kutta Method :

The formula for (4th) order Runge Kutta is given as follows :

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_i, y_i)$$

$$k_2 = h f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_i + h, y_i + k_3)$$

$$\text{Ex : } \frac{dy}{dx} = y, \quad y(0) = 1 \quad \text{find } y \text{ at } x = 1, \quad h = 1$$

Solution

$$k_1 = h f(x_i, y_i) = 1(1) = 1$$

$$k_2 = h f\left(x_i + \frac{h}{2}, y_n + \frac{k_1}{2}\right) = 1\left(1 + \frac{1}{2}\right) = 1.5$$

$$k_3 = h f\left(x_i + \frac{h}{2}, y_n + \frac{k_2}{2}\right) = 1\left(1 + \frac{1.5}{2}\right) = 1.75$$

$$k_4 = h f(x_i + h, y_n + k_3) = 1(1 + 1.75) = 2.75$$

$$y_1 = y_o + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(1 + 2(1.5) + 2(1.75) + 2.75) = 2.70833$$

$$\text{Ex : } \frac{dy}{dx} = x^2 + y, \quad y(0) = 1 \quad \text{find } y \text{ at } x = 1, \quad h = 0.5$$

Solution

$$x = 0, \quad y = 1, \quad h = 0.5$$

$$k_1 = 0.5(0^2 + 1) = 0.5$$

$$k_2 = 0.5\left[\left(0 + \frac{0.5}{2}\right)^2 + \left(1 + \frac{0.5}{2}\right)\right] = 0.65625$$

$$k_3 = 0.5\left[\left(0 + \frac{0.5}{2}\right)^2 + \left(1 + \frac{0.62625}{2}\right)\right] = 0.6953$$

$$k_4 = 0.5[(0 + 0.5)^2 + (1 + 0.6953)] = 0.97265$$

$$y_1 = 1 + \frac{1}{6}(0.5 + 2(0.65625 + 0.6953) + 0.97265) = 1.696$$

$$x = 0.5, \quad y = 1.696, \quad h = 0.5$$

$$k_1 = 0.5(0.5^2 + 1.696) = 0.973$$

$$k_2 = 0.5[(0.5 + \frac{0.5}{2})^2 + (1.696 + \frac{0.973}{2})] = 1.3725$$

$$k_3 = 0.5[(0.5 + \frac{0.5}{2})^2 + (1.696 + \frac{1.3725}{2})] = 1.4724$$

$$k_4 = 0.5[(0.5 + 0.5)^2 + (1.696 + 1.4724)] = 2.0842$$

$$y_2 = 1.696 + \frac{1}{6}(0.973 + 2(1.3725 + 1.4724) + 2.0842) = 3.1538$$

Numerical Methods in General

Taylor Series

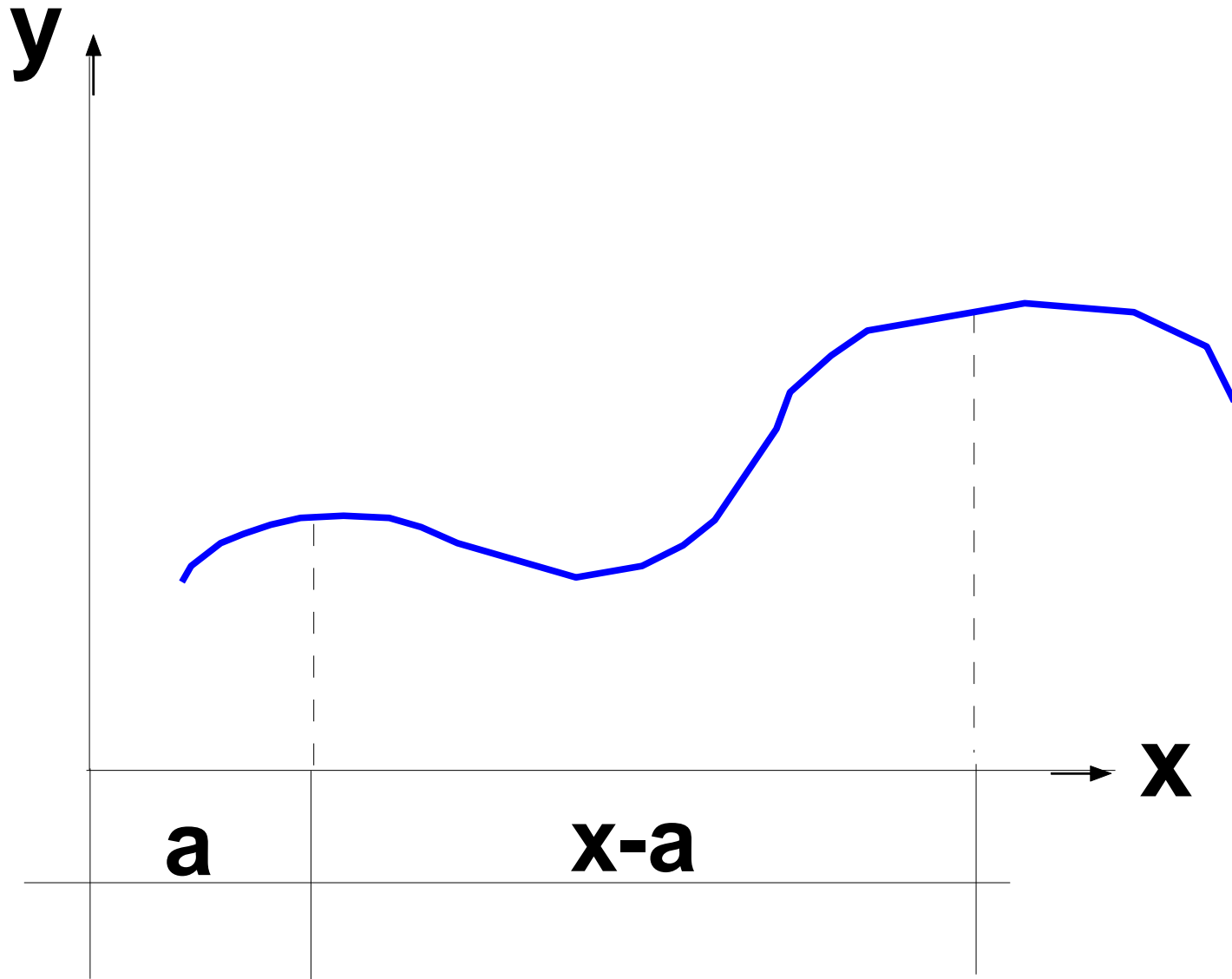
Let $f(x)$ is a series written as

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n$$

where; $a_0, a_1, a_2, \dots, a_n$ are coefficients

This is the expansion of :

$[f(x)]$ about (a)



Determination of Coefficients

1- $a_0 \rightarrow$ by put $x = a$

$$f(x) = f(a) = a_0 + a_1(a - a) + a_2(a - a)^2 + \dots + a_n(a - a)^n$$

$$\therefore a_0 = f(a)$$

2- $a_1 \rightarrow$ from $\frac{df(x)}{dx}$, put $x = a$

$$\frac{df(x)}{dx} = f'(x) = 0 + a_1 + 2a_2(x - a) + \dots$$

$$f'(a) = 0 + a_1 + 0$$

$$\therefore a_1 = f'(a)$$

$$3 - a_2 \rightarrow \text{from } \frac{d^2 f(x)}{dx^2}$$

$$\frac{d^2 f(x)}{dx^2} = f''(x) = 0 + 0 + 2a_2 + 6a_3(x - a) + \dots$$

Put $x = a$

$$f''(a) = 0 + 0 + 2a_2 + 0$$

$$\therefore a_2 = \frac{f''(a)}{2}$$

$$4 - a_3 \rightarrow \text{from } \frac{d^3 f(x)}{dx^3}$$

$$\text{So; } a_3 = \frac{f'''(a)}{6} = \frac{f'''(a)}{3!}$$

$$\therefore f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a) (x-a)^n$$

If $(a = 0)$ the expansion is about the origin.

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots + f^{(n)}(0)\frac{x^n}{n!}$$

Maclurian Series

Expansion of $[f(x + \Delta x)]$

Write; $x = a;$ $x - a = \Delta x$

$$f(x + \Delta x) = f(x) + f'(x)(\Delta x) + f''(x) \frac{(\Delta x)^2}{2!} \\ + \dots\dots\dots + f^{(n)}(x) \frac{(\Delta x)^n}{n!}$$

Ex. (1) Find $\sqrt{17}$

Solution

$$x = 16 \quad \Delta x = 17 - 16 = 1$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$$

$$\sqrt{17} = \sqrt{16} + \frac{1}{2} (16)^{-\frac{1}{2}} (1) + \left[-\frac{1}{4} (16)^{-\frac{3}{2}} \frac{(1)^2}{2!} \right] = 4.12304$$

Ex. (2) Find $\sin(28)$

Solution

$$f(x) = \sin x, \quad x = \frac{\pi}{6}; \quad \Delta x = -2 * \frac{\pi}{180}$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$\begin{aligned} \sin(28) &= \sin\left(\frac{\pi}{6}\right) + \left(\cos\left(\frac{\pi}{6}\right)\right)\left(\frac{-2\pi}{180}\right) + \left(-\sin\left(\frac{\pi}{6}\right)\right)\left(\frac{\left(\frac{-2\pi}{180}\right)^2}{2!}\right) \\ &= 0.46946 \approx 0.47 \end{aligned}$$

Expand e^x in Maclaurian (Taylor) Series

$$f(x) = e^x$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots + f^{(n)}(0)\frac{x^n}{n!}$$

$$f(x) = e^x \text{ at } x = 0 \Rightarrow f(0) = 1$$

$$f'(x) = f''(x) = f'''(x) = \dots f^{(n)}(x) = e^x$$

$$f'(0) = 1 = f''(0) = f'''(0) = \dots f^{(n)}(0)$$

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

Solution of Higher al – gebaric and Transcendental Equations :

$$- f(x) = a_o + a_1x = 0$$

$$a_o + a_1x = 0$$

$$a_1x = -a_o$$

$$\therefore x = \frac{-a_o}{a_1} \quad \text{one root}$$

$$- \text{If } f(x) = ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$f(x) = x^6 + 3x^4 - 6x + 7 = 0$$

or

$$f(x) = \sin x + x + e^x = 0$$

or

$$f(x) = \cosh x + \tan x + \ln x = 0$$

Approximation Solution used to Solve the Equations :

Newton – Raphson Method

Suppose $f(x) = 0$; find the root x ,

Start with an initial estimate $(x^{(o)})$ and improve

estimate to, $x^{(1)} = x^{(o)} - \frac{f(x^{(o)})}{f'(x^{(o)})}$ after repeat and

obtain $(x^{(2)}$ and $x^{(3)})$ and after $(k+1)$ cycles;

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

Derivation of Newton – Raphson Formula :

By Taylor Series;

Let $(x^{(o)})$ initial estimate for the root $[f(x) = 0]$

$$f(x^{(o)}) \neq 0$$

$$x^{(1)} = x^{(o)} + \Delta x^{(o)}$$

$\Delta x^{(o)} \rightarrow$ *is the correction*

$$f(x^{(o)} + \Delta x^{(o)}) = f(x^{(o)}) + f'(x^{(o)})(\Delta x^{(o)}) + \dots\dots\dots$$

$$0 = f(x^{(o)}) + f'(x^{(o)})(\Delta x^{(o)})$$

$$-f(x^{(o)}) = f'(x^{(o)})(\Delta x^{(o)})$$

$$-f(x^{(o)}) = f'(x^{(o)})(\Delta x^{(o)})$$

$$(\Delta x^{(o)}) = -\frac{f(x^{(o)})}{f'(x^{(o)})}$$

$$x^{(1)} = x^{(o)} + \Delta x^{(o)}$$

$$x^{(1)} = x^{(o)} - \frac{f(x^{(o)})}{f'(x^{(o)})}$$

By repetition :

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

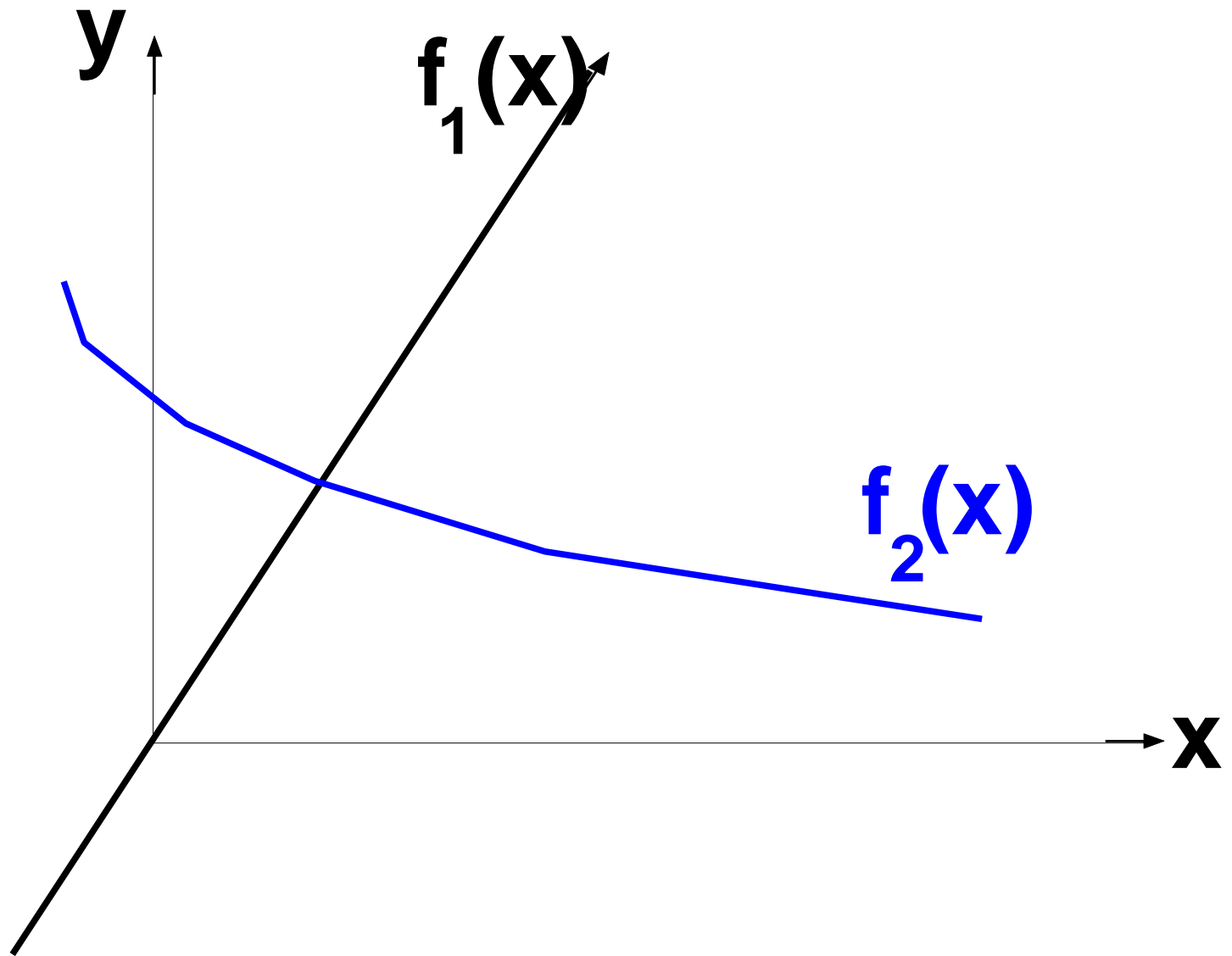
Ex1: Solve $2x - e^{-x} = 0$

Solution :

$$2x = e^{-x}$$

$$f_1(x) = 2x \quad , \quad f_2 = e^{-x}$$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$



$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$f'(x) = 2 + e^{-x}$$

$$\text{Start } x^0 = 0$$

$$x^1 = x^0 - \frac{2x^0 - e^{-x^0}}{2 + e^{-x^0}} = 0 - \frac{2(0) - e^{-0}}{2 + e^{-0}} = \frac{1}{3} = 0.333$$

$$x^2 = 0.333 - \frac{2(0.333) - e^{-0.333}}{2 + e^{-0.333}} = 0.3518$$

$$x^3 = 0.3518 - \frac{2(0.3518) - e^{-0.3518}}{2 + e^{-0.3518}} = 0.3517 \approx 0.3518$$

$$\therefore x = 0.3517$$

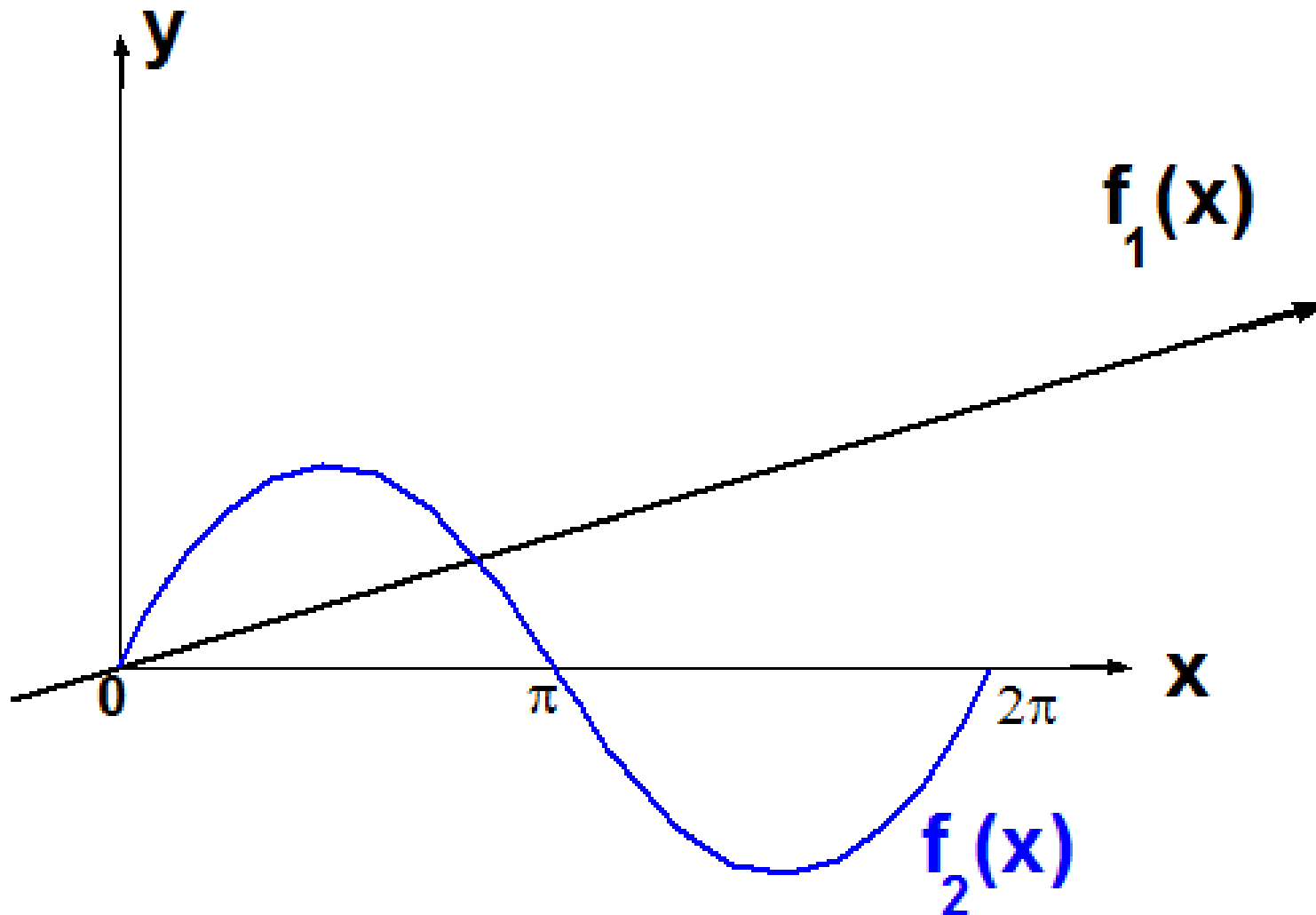
*Ex2: Use Newton – Raphson Method to solve;
 $0.25x - \sin x = 0$ (required the root > 0)*

Solution :

$$f_1(x) = 0.25x \quad , \quad f_2 = \sin x$$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

Start $x^0 = 3$



$$x^1 = x^o - \frac{f(x^o)}{f'(x^o)} = 3 - \frac{0.25(3) - \sin(3)}{0.25 - \cos(3)}$$

$$= 2.50896 \approx 2.51$$

$$x^2 = x^1 - \frac{f(x^1)}{f'(x^1)} = 2.51 - \frac{0.25(2.51) - \sin(2.51)}{0.25 - \cos(2.51)}$$

$$= 2.475 \approx 2.51$$

$$\therefore x = 2.475$$

Iteration Method

To solve $(f(x) = 0)$, transform $(f(x) = 0)$ to the form of $(x = g(x))$. If (x^k) an estimate for (x) will improve as;

$$x^{k+1} = g(x)^k$$

The estimation is convergence when;

$$\left| g'(x)^k \right| < 1$$

Ex1 : Solve $4x - 6 = 0$

Solution :

$$x + 3x = 6$$

$$x = 6 - 3x$$

$$x = g(x) = 6 - 3x$$

$$g'(x) = -3$$

$|-3| > 1$ *No convergence*

$$\text{or : } x + 3x = 6$$

$$3x = 6 - x$$

$$x = \frac{1}{3} (6 - x)$$

$$g(x) = \frac{1}{3} (6 - x)$$

$$g'(x) = -\frac{1}{3}$$

$$\left| -\frac{1}{3} \right| = \frac{1}{3} < 1$$

$\therefore o.k.$

$$x = \frac{1}{3}(6 - x)$$

$$x^o = 0$$

$$x^1 = \frac{1}{3}(6 - 0) = 2$$

$$x^2 = \frac{1}{3}(6 - 2) = 1.333$$

$$x^3 = \frac{1}{3}(6 - 1.333) = 1.556$$

$$x^4 = \frac{1}{3}(6 - 1.556) = 1.481$$

$$x^5 = \frac{1}{3}(6 - 1.481) = 1.506$$

$$x^6 = \frac{1}{3}(6 - 1.506) = 1.498$$

$$x^7 = \frac{1}{3}(6 - 1.498) = 1.5006$$

$$\therefore x = 1.5006$$

Ex1 : Solve $x^2 - e^{-x} = 0$

Solution :

$$x^2 = e^{-x}$$

$$x = e^{-0.5x}$$

$$g(x) = e^{-0.5x}$$

$$g'(x) = -0.5e^{-0.5x}$$

$$x^o = 0$$

$$g'(x) = -0.5e^{-0.5(0)} = -0.5$$

$$|-0.5| < 1 \quad \therefore o.k. \quad \text{Cognv.}$$

$$\text{Put } x^o = 0$$

$$x^{k+1} = g(x)^k$$

$$x^1 = e^{-0.5(0)} = 1$$

$$x^2 = e^{-0.5(1)} = 0.6065$$

$$x^3 = e^{-0.5(0.6065)} = 0.738$$

$$x^4 = e^{-0.5(0.738)} = 0.691$$

$$x^5 = e^{-0.5(0.691)} = 0.708$$

$$x^6 = e^{-0.5(0.708)} = 0.702$$

$$x^7 = e^{-0.5(0.702)} = 0.704 \approx 0.702$$

$$\therefore x = 0.702$$