$$Interpolation$$

$$From\ \ test,\ a\,set\ \ of\ \ data\ are\ \ obtained\,;$$

$$(x_o, y_o), (x_1, y_1), (x_2, y_2), \ldots\ldots\ldots\ldots\ldots\ldots (x_n, y_n)$$

$$(with\ \ n+1\ po\mathrm{int}\,s),\ construct\ \ a\ \ polynomial$$

$$of\ \ \deg ree\,(n)\ \ to\ \ pass\ \ through\ \ the\ \ above\ po\mathrm{int}\,s:$$

$$f(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\ldots + a_n x^n$$

$$The\ \ coefficien\,ts\ (a_o,\ a_1,\ a_2,\ \ldots\ldots\ldots\ldots\ldots a_n)\ are$$

$$obtained\ \ \ from\ \ the\ \ data\ \ above;$$

$$(x_o, y_o),\ (x_1, y_1),\ (x_2, y_2), \ldots\ldots\ldots\ldots\ldots (x_n, y_n)$$

$$(x_o, y_o);$$

$$y_o = a_o + a_1 x_o + a_2 x_o^2 + a_3 x_o^3 + \ldots\ldots\ldots\ldots + a_n x_o^n \quad \ldots\ldots(1)$$

$$(x_1, y_1);$$

$$y_1 = a_o + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + \ldots\ldots\ldots\ldots + a_n x_1^n \quad \ldots\ldots(2)$$

$$(x_2, y_2);$$

$$y_2 = a_o + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3 + \ldots\ldots\ldots\ldots + a_n x_2^n \quad \ldots\ldots(3)$$

$$\vdots$$

$$\vdots$$

$$(x_n, y_n);$$

$$y_n = a_o + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \ldots\ldots\ldots\ldots + a_n x_n^n \quad \ldots\ldots(n+1)$$

$1-\ No.\ of\ \text{variables}\ (n+1)\ (a_o,\ a_1,\ a_2, \ldots\ldots\ldots a_n)$

$2-\ No.\ of\ equations\ (n+1)$

$3-Solve\ by\ (Gauss-Elimination),\ as\ example$

$Example: The\ following\ data\ are\ obtained\ from\ test:$

$x_i : 0 \quad 1 \quad 3 \quad 4$

$y_i : 1 \quad 2 \quad 10 \quad 17$

$\text{Required}: i)\ Construct\ a\ polynomial\ equation$

$passing\ through\ the\ points$

$ii)\ f(x)\ at\ x = 2$

Solution :

$$f(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3$$

$(0,1);$

$$1 = a_o + a_1(0) + a_2(0)^2 + a_3(0)^3 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$(1,2);$

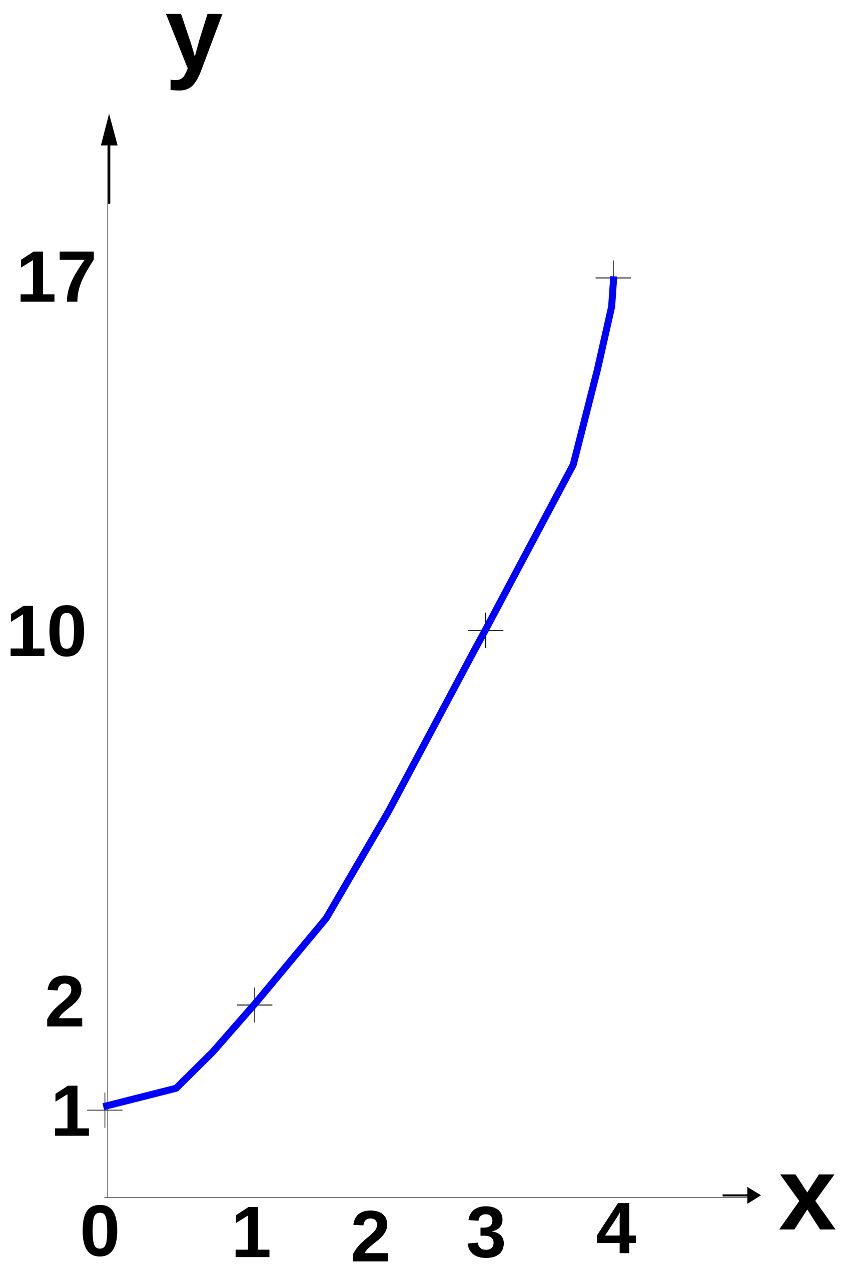
$$2 = a_o + a_1(1) + a_2(1)^2 + a_3(1)^3 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$(3,10);$

$$10 = a_o + a_1(3) + a_2(3)^2 + a_3(3)^3 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$(4,17);$

$$17 = a_o + a_1(4) + a_2(4)^2 + a_3(4)^3 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$


y
17
10
2
1
0
1
2
3
4
x

$$a_o = 1,$$

$$a_1 + a_2 + a_3 = 1$$

$$3a_1 + 9a_2 + 27a_3 = 9$$

$$4a_1 + 16a_2 + 64a_3 = 16$$

$$Solve\ the\ \ above\ equations;\ get;$$

$$a_1 = 0,\ \ a_2 = 1\ \ \ and\ \ \ a_3 = 0$$

$$a_1 = 0, \quad a_2 = 1 \quad and \quad a_3 = 0$$

$$\therefore y = f(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3$$

$$= 1 + x^2$$

$$At \quad x = 2 \Rightarrow f(x) = 1 + (2)^2 = 5$$

Differences of Interpolating polynomials:

1 – *Newton forward Interpolting:*

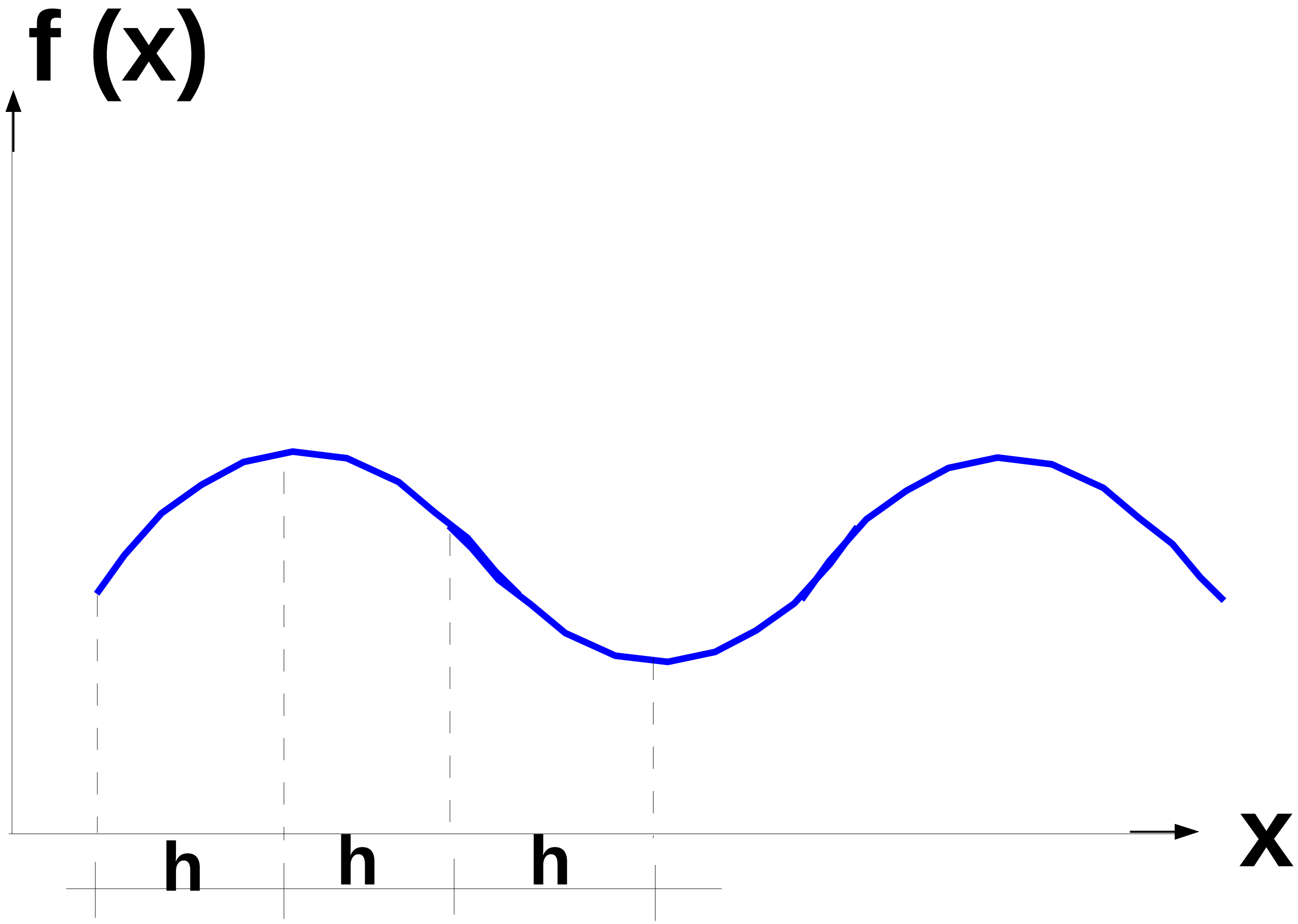
Suppose that a table relating a dependent variable $f(x)$ to an independent variable x is given as:

$$x_i : \quad x_o \qquad x_1 \qquad x_2 \ldots\ldots\ldots\ldots\ldots\ldots x_n$$

$$f(x_i) : f(x_o) \qquad f(x_1) \qquad f(x_2) \ldots\ldots\ldots\ldots f(x_n)$$

$$\Delta f x_i = f_{i+1} - f_i$$

$$h = x_{i+1} - x_i \quad (equal \;\; intervals)$$



x_i	$f(x_i)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x_0	$f(x_0)$				
		Δf_0			
x_1	$f(x_1)$		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	$f(x_2)$		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	$f(x_3)$		$\Delta^2 f_2$		
		Δf_3			
x_4	$f(x_4)$				

$$f(x+h) = Ef(x_i)$$

$$E = Shift \quad operator$$

$$f(x+2h) = E^2 f(x_i)$$

$$f(x+\alpha h) = E^\alpha f(x_i)$$

$$E = 1 + \Delta$$

$$\alpha = \frac{x - x_i}{h}$$

$$f(x+\alpha h) = E^\alpha f(x_i) = (1+\Delta)^\alpha f(x_i)$$

$$By\ bionomial\ \ formula:$$

$$f(x+\alpha h)=[1+\Delta\alpha+\Delta^{2}\frac{\alpha(\alpha-1)}{2!}+$$

$$\Delta^{3}\frac{\alpha(\alpha-1)(\alpha-2)}{3!}+.......]f(x_i)$$

$2-Newton \;\; Backward \;\; Interpolting:$

$$E = 1 - \Delta$$

$$f(x+\alpha h) = [1 + \Delta\alpha + \Delta^2 \frac{\alpha(\alpha+1)}{2!} + \Delta^3 \frac{\alpha(\alpha+1)(\alpha+2)}{3!} + \ldots\ldots] f(x_i)$$

$Example : Given\ the\ following\ data, approximate$

$the\ functional\ value\ at\ (x = 1.5)$

x_i :	0	1	2	3	4
y_i :	-1	0	3	8	15

Solution

x_i	$f(x_i)$	Df	D^2f	D^3f
0	-1			
		1		
1	0		2	
		3		0
2	3		2	
		5		0
3	8		2	
		7		
4	15			

$$1-Forward: \quad x=1.5 \qquad h=1$$

$$\alpha=\frac{x-x_i}{h} \;; \quad x_i=0$$

$$\alpha=\frac{1.5-0}{1}=1.5$$

$$f(x+\alpha h)=f(1.5)=[1+\alpha\Delta+\Delta^2\frac{\alpha(\alpha-1)}{2!}]\,f(x_i)$$

$$f(x)=-1, \quad \Delta f(x)=1 \quad and \quad \Delta^2 f(x)=2$$

$$f(1.5)=[-1+1.5*1+2*\frac{1.5(1.5-1)}{2}]=0.5+0.75=1.25$$

$$1-Backward: \quad x = 1.5 \qquad h = 1 \qquad x_i = 3$$

$$\alpha = \frac{x - x_i}{h} \quad ;$$

$$\alpha = \frac{1.5 - 3}{1} = -1.5$$

$$At \quad x = 3$$

$$f(x) = 8, \quad \Delta f(x) = 5 \quad and \quad \Delta^2 f(x) = 2$$

$$f(1.5) = [1 + \alpha$$

$$f(1.5) = [1 + \Delta\alpha + \Delta^2 \frac{\alpha(\alpha+1)}{2!}] f(x_i)$$

$$f(1.5) = [8 + 5*(-1.5) + 2*\frac{-1.5(-1.5+1)}{2}]$$

$$= 8 + 7.5 + 0.75 = 1.25$$

$3-Lagrange\ Interpolation$

$Consider\ (x_o, y_o),(x_1, y_1),(x_2, y_2)..........(x_n, y_n)$

$Lagrange\ Interpolation\ Polynomial\ is:$

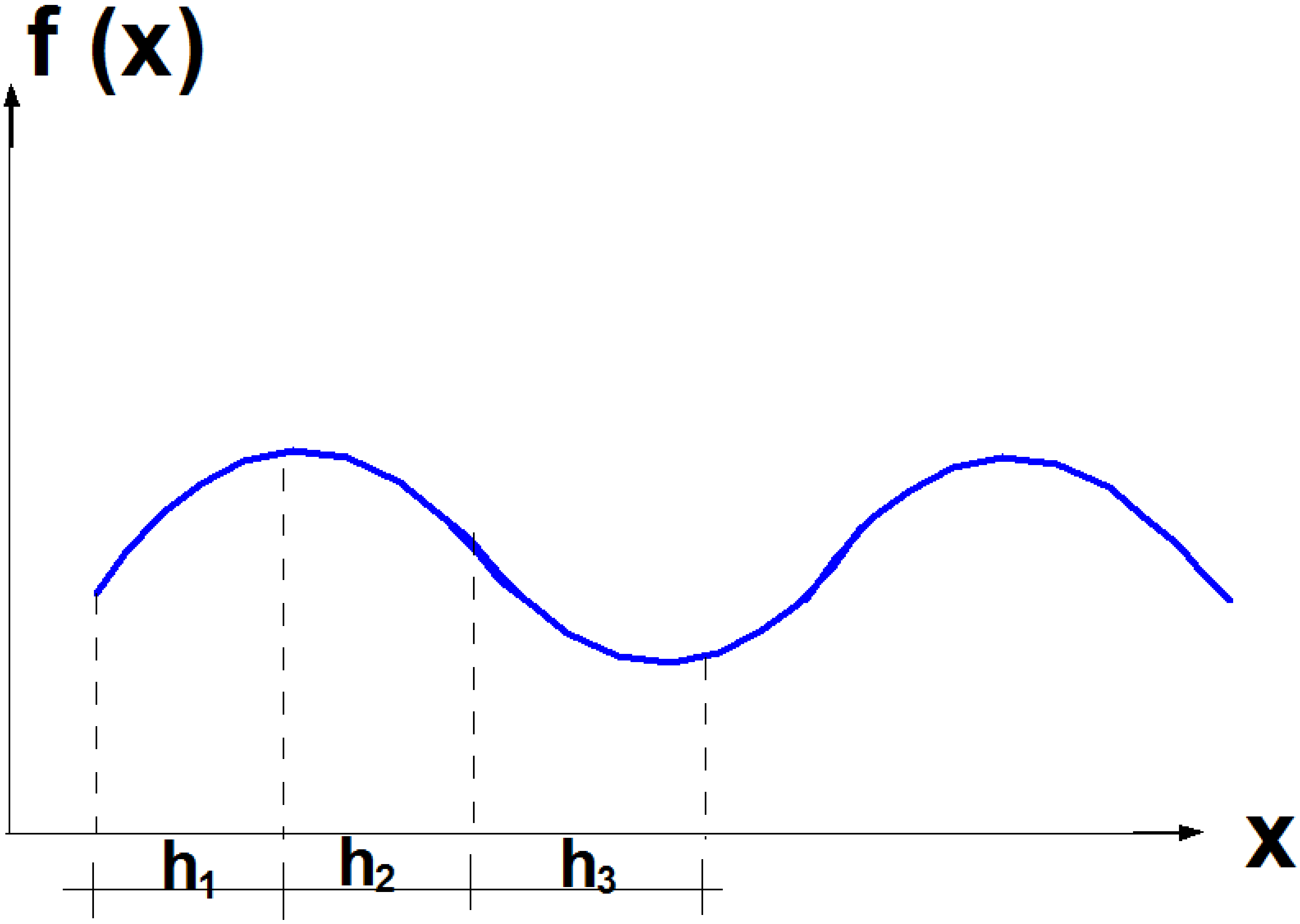
$$f(x)=Ln(x)=\sum_{k=0}^{k=n}\frac{\ell_k(x)}{\ell_k(x_k)}y_k$$

$$\ell_o=(x-x_1)(x-x_2)(x-x_3).......$$

$$\ell_1=(x-x_o)(x-x_2)(x-x_3).......$$

$$\vdots$$

$$\ell_3=(x-x_o)(x-x_1)(x-x_2)(x-x_4).......$$


f (x)
x
h1
h2
h3

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)....}{(x_o-x_1)(x_o-x_2)(x_o-x_3).......}*y_o+$$

$$\frac{(x-x_o)(x-x_2)(x-x_3)....}{(x_1-x_o)(x_1-x_2)(x_1-x_3).......}*y_1+$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)....}{(x_2-x_o)(x_2-x_1)(x_2-x_3).......}*y_2+.....$$

$$(x=x_o)\Rightarrow f(x)=y_o \quad ;(x=x_1)\Rightarrow f(x)=y_1$$

$Or:$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)....(x-x_n)}{(x_o-x_1)(x_o-x_2)(x_o-x_3)....(x_o-x_n)} * y_o +$$

$$\frac{(x-x_o)(x-x_2)(x-x_3)...(x-x_n)}{(x_1-x_o)(x_1-x_2)(x_1-x_3)....(x_1-x_n)} * y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)....(x-x_n)}{(x_2-x_o)(x_2-x_1)(x_2-x_3)....(x_2-x_n)} * y_2 + ...$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)....(x-x_{n-1})}{(x_n-x_o)(x_n-x_1)(x_n-x_3)....(x_n-x_{n-1})} * y_n$$

$Example : Given$

$$
\begin{array}{lcccc}
x_i : & 0 & 2 & 3 & 5 \\
y_i : & 1 & 7 & 13 & 31
\end{array}
$$

$\text{Required} : \; f(x) \;\; at \;\; x = 4$

$$y = f(4) = \frac{(4-2)(4-3)(4-5)}{(0-2)(0-3)(0-5)} * 1 + \frac{(4-0)(4-3)(4-5)}{(2-0)(2-3)(2-5)} * 7$$

$$+ \frac{(4-0)(4-2)(4-5)}{(3-0)(3-2)(3-5)} * 13 + \frac{(4-0)(4-2)(4-3)}{(5-0)(5-2)(5-3)} * 31$$

$$= \frac{2*1*(-1)}{-2*(-3)(-5)} * 1 + \frac{4*1*(-1)}{2*(-1)(-3)} * 7 + \frac{4*2*(-1)}{3*1*(-2)} * 13 + \frac{4*2*1}{5*3*2} * 31$$

$$= \frac{-2}{-30} * 1 + \frac{-4}{6} * 7 + \frac{-8}{-6} * 13 + \frac{8}{30} * 31 = 21$$

$$Example : Given$$

x_i :	1	2	3	5
y_i :	0	3	8	14

$$\text{Re}\,quired : \; f(x) \;\; at \;\; x = 4$$

$$y = f(4) = \frac{(4-2)(4-3)(4-5)}{(1-2)(1-3)(1-5)} * 0 + \frac{(4-1)(4-3)(4-5)}{(2-1)(2-3)(2-5)} * 3$$

$$+ \frac{(4-1)(4-2)(4-5)}{(3-1)(3-2)(3-5)} * 8 + \frac{(4-1)(4-2)(4-3)}{(5-1)(5-2)(5-3)} * 24$$

$$= 0 + \frac{3*1*(-1)}{1*(-1)(-3)} * 3 + \frac{3*2*(-1)}{2*1*(-2)} * 8 + \frac{3*2*1}{4*3*2} * 24$$

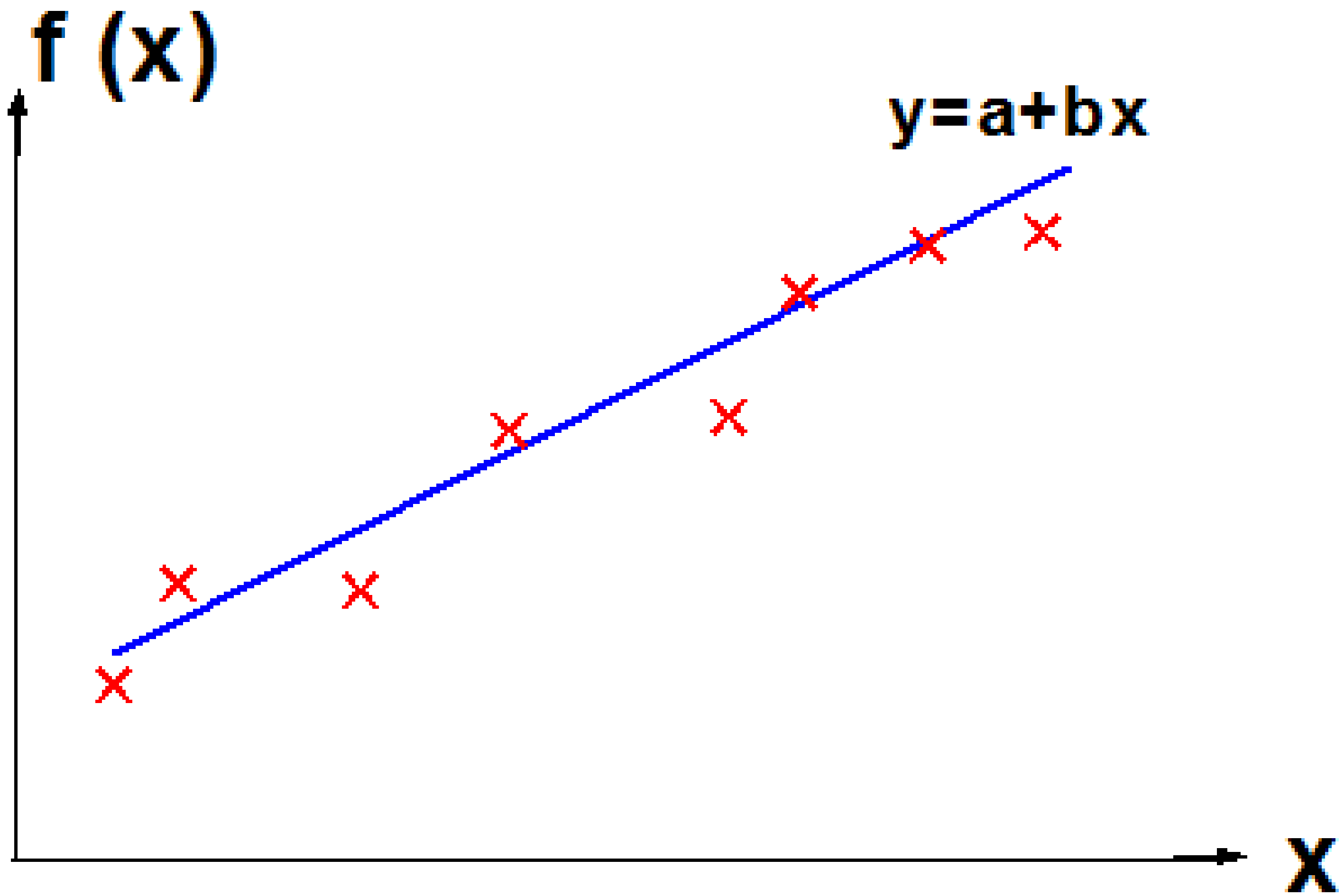
$$= \frac{-3}{3} * 3 + \frac{-6}{-4} * 8 + \frac{6}{24} * 24 = 15$$

Curve Fitting

The method of Least squares

Consider a sample of $(x_1, y_1),(x_2, y_2),(x_3, y_3)...(x_n, y_n)$ of size (n). It is rwquired to draw a best line passing the points above, such that the sum of the square errors measured in $(y - direction)$ is minimum.

We shall study the best straight line through the given points.


f (x)
y=a+bx
x

$Suppose\ that\ the\ best\ line\ is:$

$$y = a + bx$$

$$e_i = (error\ at\ x = x_i) = y_i - (a + bx_i) = y_i - y$$

$$\sum e_i = \sum [y_i - (a + bx_i)]$$

$$\sum_{i=1}^{n} (e_i)^2 = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

$$\sum_{i=1}^{n} (e_i)^2 = S$$

$$\therefore S = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

$$S = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

$$\frac{\partial S}{\partial a} = 2[\sum_{i=1}^{n} \{y_i - (a + bx_i)\}](-1) = 0$$

$$\sum_{i=1}^{n} [y_i - (a + bx_i)] = 0$$

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} (a + bx_i) = 0$$

$$\therefore \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (a + bx_i)$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} a + \sum_{i=1}^{n} bx_i = a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i$$

$$\therefore \sum_{i=1}^{n} y_i = an + b\sum_{i=1}^{n} x_i \text{............}(1)$$

$$S=\sum_{i=1}^{n}[y_i-(a+bx_i)]^2$$

$$\frac{\partial S}{\partial b}=2[\sum_{i=1}^{n}\{y_i-(a+bx_i)\}](-x_i)=0$$

$$\sum_{i=1}^{n}[y_i-(a+bx_i)]x_i=0$$

$$\sum_{i=1}^{n}y_ix_i-\sum_{i=1}^{n}(a+bx_i)x_i=0$$

$$\therefore\sum_{i=1}^{n}y_ix_i=\sum_{i=1}^{n}(a+bx_i)x_i$$

$$\sum_{i=1}^{n}y_ix_i=\sum_{i=1}^{n}ax_i+\sum_{i=1}^{n}bx_i^2$$

$$\therefore\sum_{i=1}^{n}y_i=a\sum_{i=1}^{n}x_i+b\sum_{i=1}^{n}x_i^2\ldots\ldots\ldots\ldots(2)$$

$$\therefore \sum_{i=1}^{n} y_i = an + b\sum_{i=1}^{n} x_i \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore \sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \ldots\ldots\ldots\ldots(2)$$

$$The\ Eqs.\ (1)\ and\ (2)\ can\ be\ written\ in\ matrix\ form:$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i x_i \end{bmatrix}$$

Example : Required the best straight line passing the points :

x_i :	0	1	3	4
y_i :	1	3	7	10

Solution :

x_i	y_i	$y_i x_i$	x_i^2
0	1	0	0
1	3	3	1
3	7	21	9
4	10	40	16
$\sum$ 8	21	64	26

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 21 \\ 64 \end{bmatrix}$$

$$4a + 8b = 21$$

$$8a + 26b = 64$$

$$Solve\ the\ \ above\ \ Eqs.\ getting\ \ \ a = 0.85\ \&\ b = 2.2$$

$$y = a + bx$$

$$\therefore\ y = 0.85 + 2.2x$$

Example: Fit the data shown below with a best line passing through the origin: $[y = bx]$.

x_i:	1	2	3	5
y_i:	2	5	7	9

$Solution:$

	x_i	y_i	$y_i x_i$	x_i^2
	1	2	2	1
	2	5	10	4
	3	7	21	9
	5	9	45	25
$\sum$	11	23	78	39

$$S=\sum_{i=1}^{n}[y_i-bx_i]^2$$

$$\frac{\partial S}{\partial b}=\sum_{i=1}^{n}[y_i-bx_i]\,(-2x_i)$$

$$\sum_{i=1}^{n}y_ix_i=b\sum_{i=1}^{n}x_i^2$$

$$\therefore b=\frac{\sum_{i=1}^{n}y_ix_i}{\sum_{i=1}^{n}x_i^2}=\frac{78}{39}=2$$

$$\therefore y=2x$$

Example: Find the best curve ($y = ae^{bx}$) *passing through the points:*

x_i:	0	1	3	6
y_i:	2	3	9	40

$$Solution: By\ taking\ Log.\ of\ both\ sides:$$

$$\ln y = \ln ae^{bx} = \ln a + \ln e^{ax}$$

$$Y = A + bx \quad such\ that\ (Y = \ln y\ \&\ A = \ln a)$$

	x_i	y_i	$Y_i = \ln y_i$	$x_i Y_i$	x_i^2
	0	2	0.693	0	0
	1	3	1.0986	1.0986	1
	3	9	2.197	6.591	9
	6	40	3.689	22.134	36
$\sum$	10		7.677	29.8236	46

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} * \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum x_i Y_i \end{bmatrix}$$

$$\begin{bmatrix} 4 & 10 \\ 10 & 46 \end{bmatrix} * \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7.677 \\ 29.8236 \end{bmatrix}$$

$$Solve\ the\ \ above\ \ Eqs.\ getting\ \ \ A = 0.6536 = \ln\ a \Rightarrow a = 1.9225$$

$$\&\ b = 0.5064$$

$$\therefore\ y = 1.9225\, e^{0.5064x}$$

In the case of Quadratic Parabola:

$$y = a_o + a_1 x + a_2 x^2$$

$$e_i = y_i - (a_o + a_1 x_i + a_2 x_i^2)$$

$$\sum_{i=1}^{n} (e_i)^2 = S = \sum_{i=1}^{n} [y_i - (a_o + a_1 x_i + a_2 x_i^2)]^2$$

$$\frac{\partial S}{\partial a_o} = 0 \Rightarrow \sum_{i=1}^{n} y_i = n a_o + a_1 \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_i^2 \;\ldots\ldots\ldots\ldots(1)$$

$$\frac{\partial S}{\partial a_1} = 0 \Rightarrow \sum_{i=1}^{n} x_i y_i = a_o \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + a_2 \sum_{i=1}^{n} x_i^3 \ldots.(2)$$

$$\frac{\partial S}{\partial a_2} = 0 \Rightarrow \sum_{i=1}^{n} x_i^2 y_i = a_o \sum_{i=1}^{n} x_i^2 + a_1 \sum_{i=1}^{n} x_i^3 + a_2 \sum_{i=1}^{n} x_i^4 \ldots.(3)$$

$$In\ general:$$

$$y = a_o + a_1 x + a_2 x^2 + \dots\dots\dots + a_n x^n$$

$$\frac{\partial S}{\partial a_o} = 0 \qquad \dots\dots\dots\dots(1)$$

$$\frac{\partial S}{\partial a_1} = 0 \qquad \dots\dots\dots\dots(2)$$

$$\frac{\partial S}{\partial a_2} = 0 \qquad \dots\dots\dots\dots(3)$$

$$\vdots$$

$$\vdots$$

$$\frac{\partial S}{\partial a_n} = 0 \qquad \dots\dots\dots\dots(n+1)$$