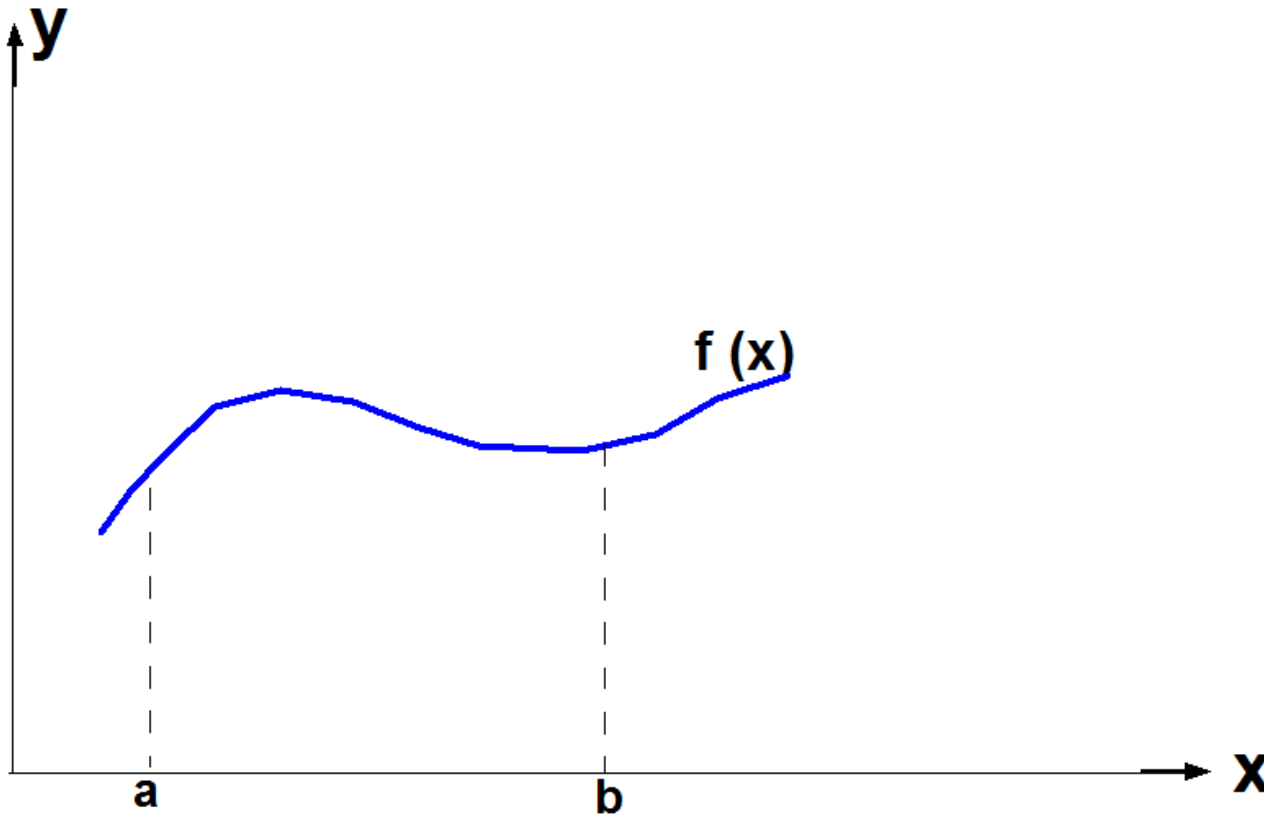


Numerical Integration

$\int \frac{\sin x}{x} dx$; $\int f(x) dx$ is not always calculated by direct integration, or $f(x)$ is given in the form of a table;

$$(x_o, y_o), (x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$$

$\int_a^b f(x) dx$ is the area under the curve
from $x = a$ to $x = b$

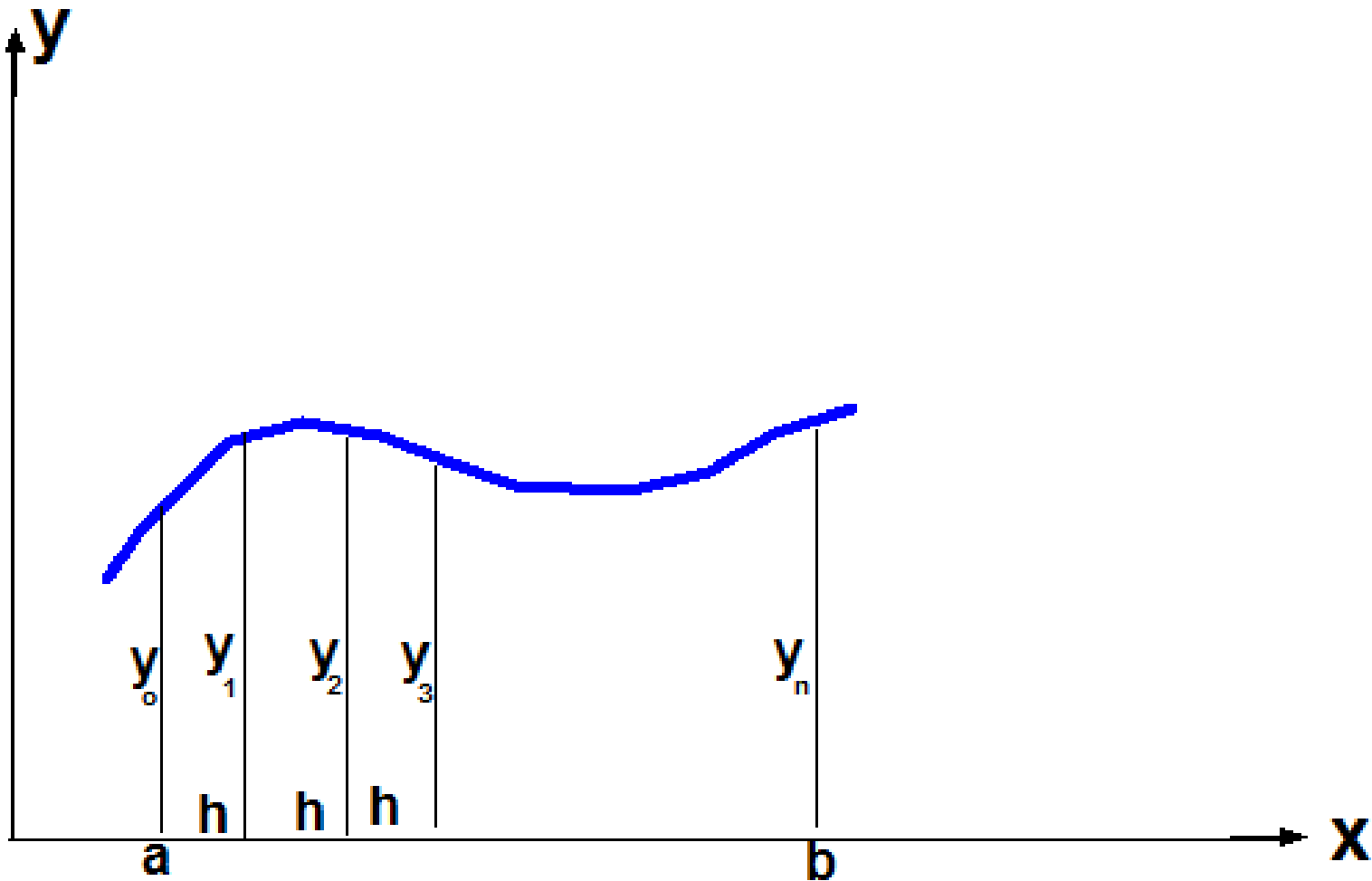


Methods of Calculated the area under the curve:

** Divide the distance $(b - a)$ into (n) steps,
the length of each step is (h) ;*

$$h = \frac{b - a}{n}$$

** Draw coordinates $(y_0, y_1, y_2, \dots, y_n)$*

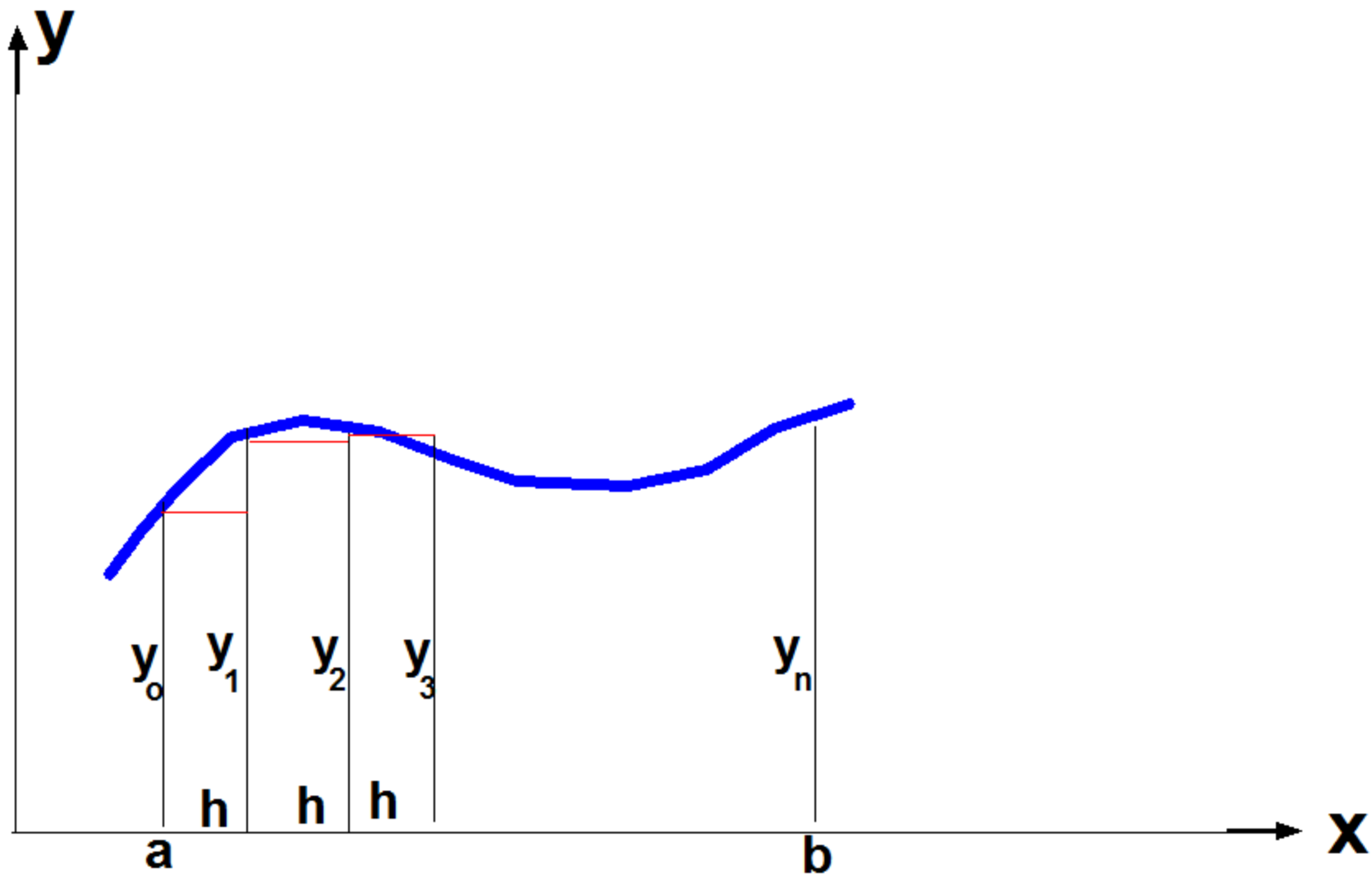


The methods are:

1– The method of rectangles :

$$I = \int_a^b f(x) \, dx$$

$$= h[y_0 + y_1 + y_2 + \dots \dots \dots y_{n-1}]$$



2– The method of trapizoids :

$$I = \int_a^b f(x) \, dx$$

$$I = A_1 + A_2 + \dots\dots\dots$$

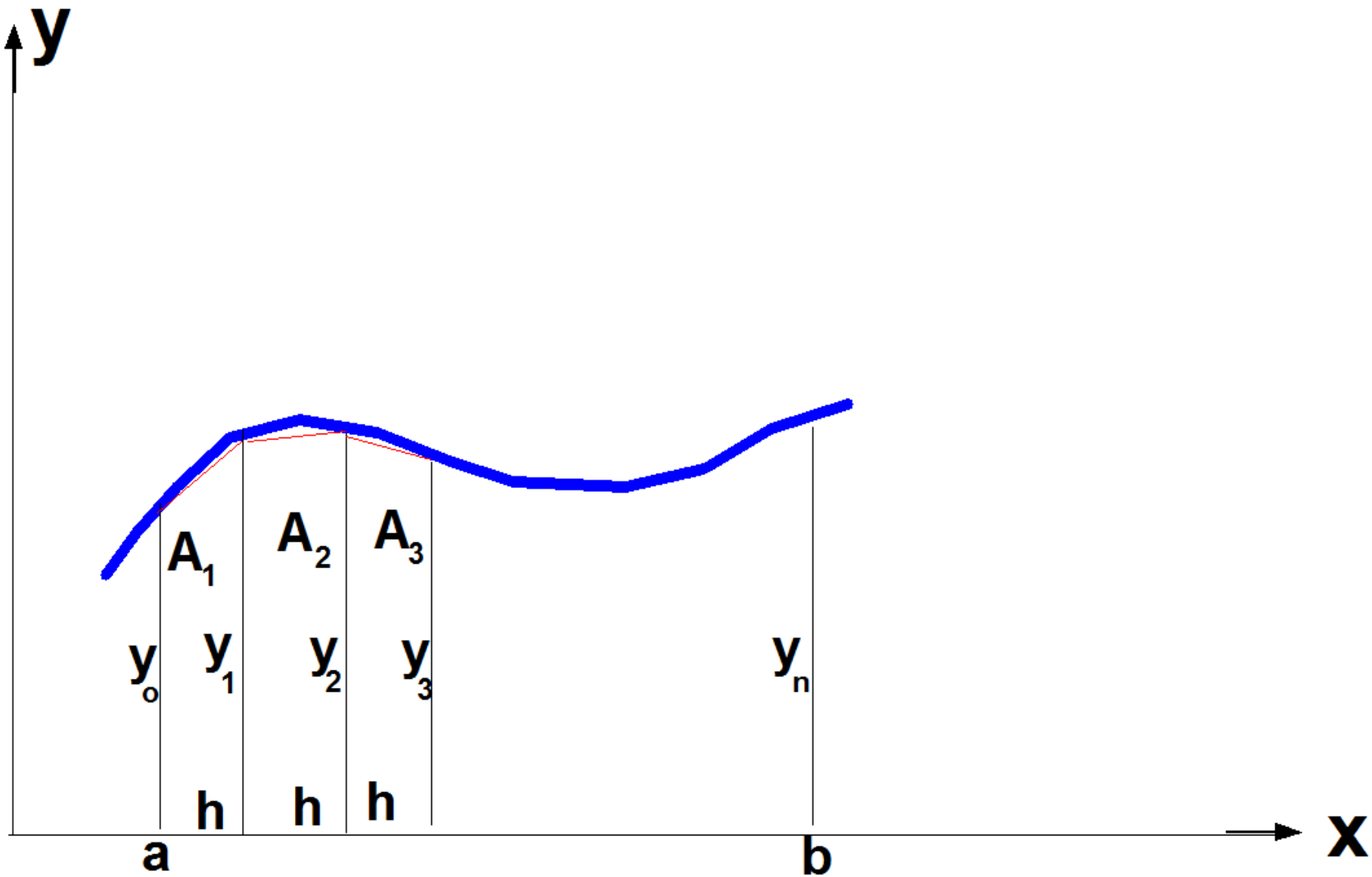
$$A_1 = h * \frac{y_o + y_1}{2}; \quad A_2 = h * \frac{y_1 + y_2}{2}; \quad A_3 = h * \frac{y_2 + y_3}{2}$$

$$\vdots$$

$$A_n = h * \frac{y_{n-1} + y_n}{2}$$

$$\therefore I = \frac{h}{2} [y_o + 2y_1 + 2y_2 + 2y_3 + \dots\dots\dots y_n]$$

$$= \frac{h}{2} [y_o + 2(y_1 + y_2 + y_3 + \dots\dots\dots) + y_n]$$



3– Simpson's Rule :

The quadratic equation is assumed to pass through the points. Thus the area into even number points.

$$A_1 = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

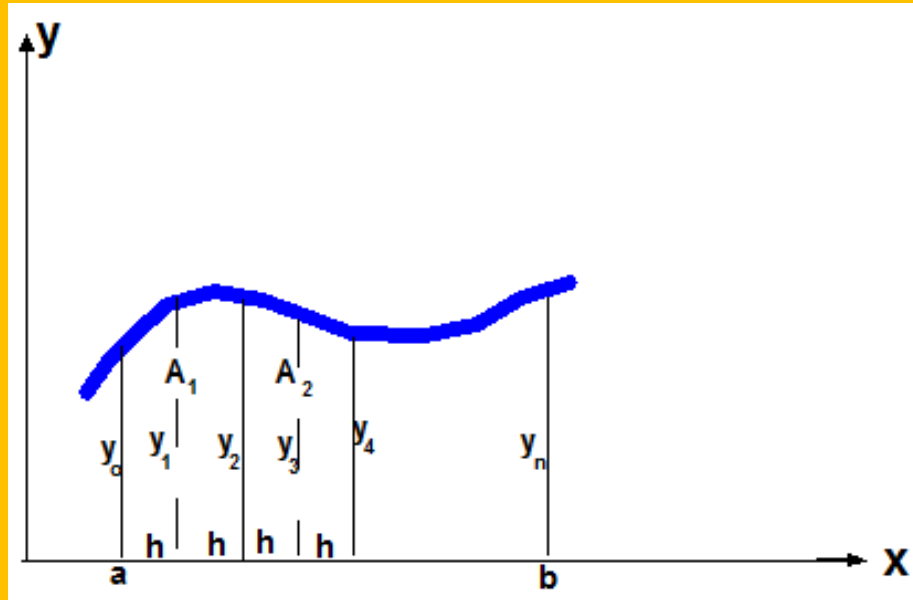
$$A_2 = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$A_3 = \frac{h}{3}[y_4 + 4y_5 + y_6]$$

\vdots

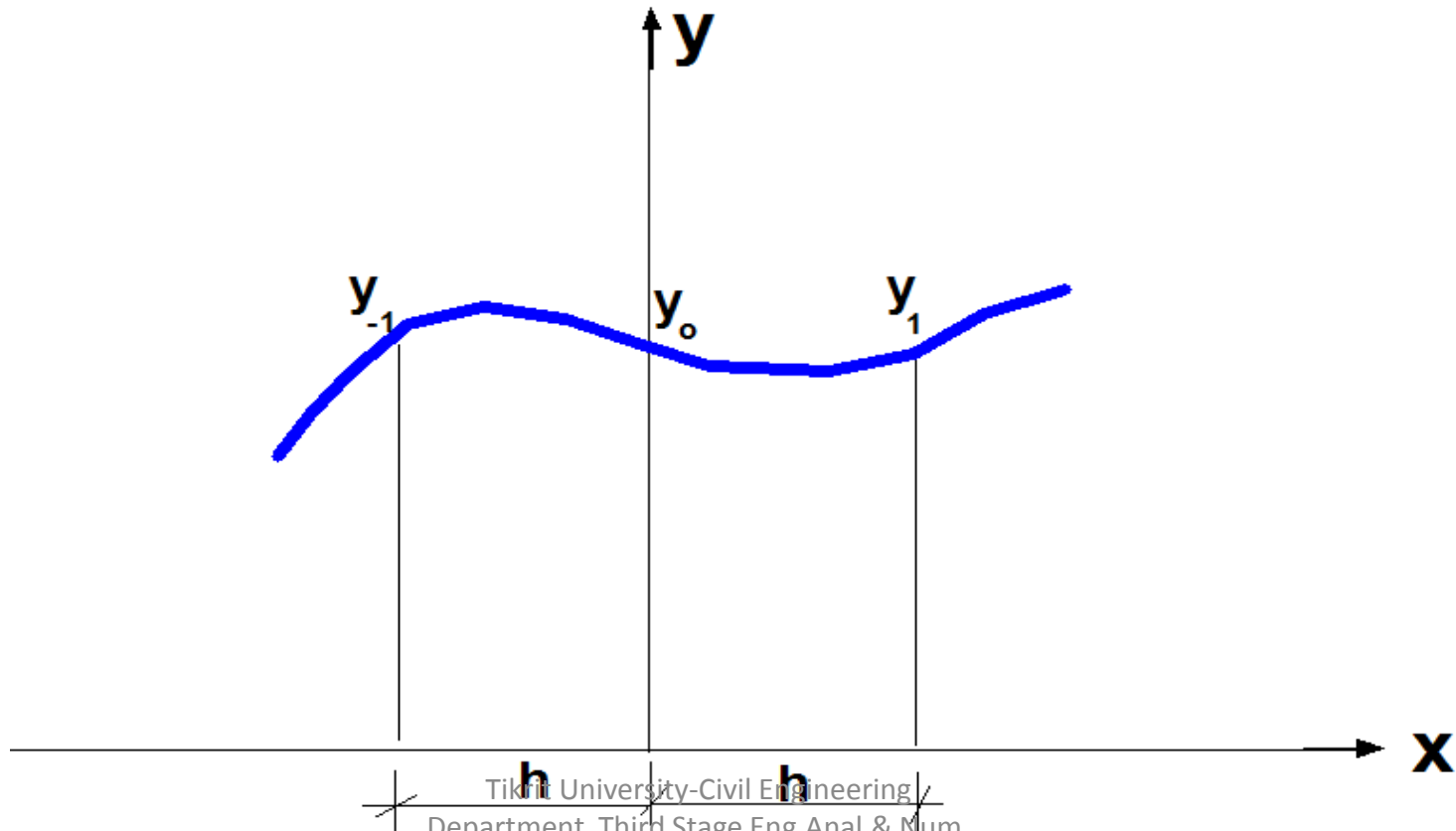
$$A = I = A_1 + A_2 + A_3 + \dots$$

$$= \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + y_6 + \dots) + y_n]$$



Assume a curve of degree (2), Quadratic equation pass through three points:

$[(-h, y_{-1}), (0, y_o) \text{ \& } (h, y_1)]$.



$$y = a_o + a_1x + a_2x^2$$

$$\text{At } x = 0 \Rightarrow y_o = a_o + a_1(0) + a_2(0)^2$$

$$\therefore a_o = y_o$$

$$(-h, y_{-1}) \Rightarrow y_{-1} = a_o + a_1(-h) + a_2(-h)^2 \dots\dots (1)$$

$$(h, y_1) \Rightarrow y_1 = a_o + a_1(h) + a_2(h)^2 \dots\dots (2)$$

$$y_{-1} = y_o - ha_1 + h^2a_2$$

$$y_1 = y_o + ha_1 + h^2a_2$$

Solve the above equations getting :

$$a_1 = \frac{y_1 - y_{-1}}{2h} \quad \text{and} \quad a_2 = \frac{y_{-1} - 2y_o + y_1}{2h^2}$$

$$\therefore y = y_o + \frac{y_1 - y_{-1}}{2h} x + \frac{y_{-1} - 2y_o + y_1}{2h^2} x^2$$

$$I = \int_{-h}^h \left[y_o + \frac{y_1 - y_{-1}}{2h} x + \frac{y_{-1} - 2y_o + y_1}{2h^2} x^2 \right] dx$$

After integration and substitution get :

$$A = I = \frac{h}{3} [y_{-1} + 4y_o + y_1]$$

Example (1) : Evaluate $\int_0^1 \frac{\sin x}{x} dx$ use $n = 4$ panels.

Solution : $h = \frac{1-0}{4} = 0.25$

$x :$	0	0.25	0.5	0.75	1.0
$f(x) :$	1	0.99	0.96	0.91	0.84

(1) *By rectangular method :*

$$I = 0.25[1 + 0.99 + 0.96 + 0.91] = 0.965$$

OR :

$$I = 0.25[0.99 + 0.96 + 0.91 + 0.84] = 0.925$$

Avg. = 0.945 (from center of points)

(2) *By trapezoidal :*

$$I = \frac{0.25}{2}[1 + 2(0.99 + 0.96 + 0.91) + 0.84] = 0.945$$

(4) *By Simpson's rule :*

$$I = \frac{0.25}{3} [1 + 4(0.99 + 0.91) + 2(0.96) + 0.84] = 0.9467$$

(2) *By Taylor Series :*

$$I = \int_0^1 \frac{\sin x}{x} dx$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

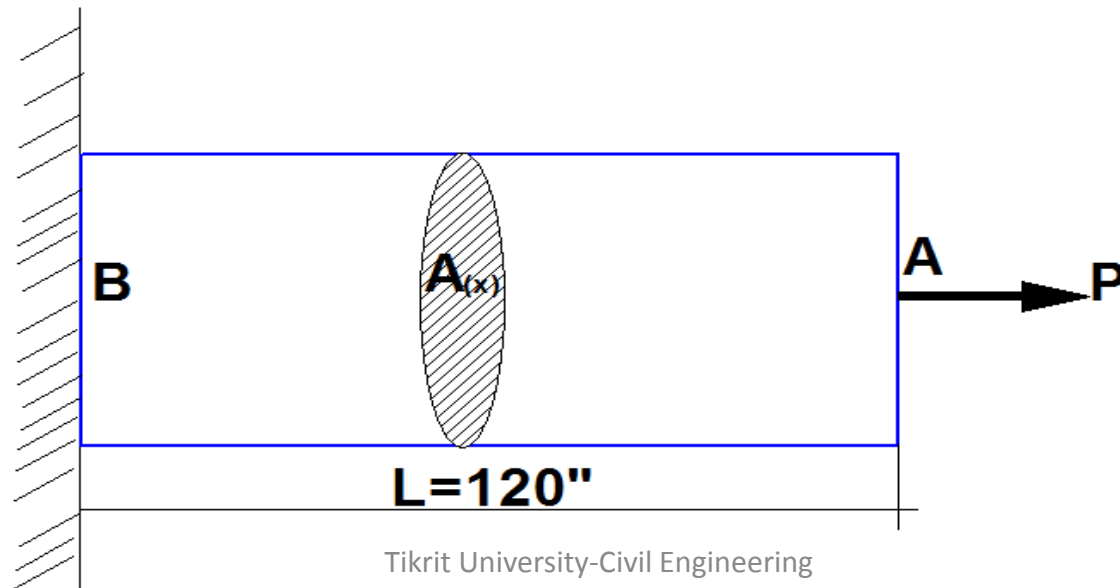
$$I = \int_0^1 \left[\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \right) / x \right] dx$$

$$= \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} \right) dx = 0.946$$

Example (2): Determine the relative deflection of point (A) with respect to point (B).

$$P_{(x)} = 10 \text{ kips}, \quad E = 30 * 10^3 \text{ ksi},$$

$$A = \left(-\frac{1}{3600} x^2 + \frac{1}{30} x + 1 \right) \text{ in}^2$$



Solution :

$$d\delta = \frac{P_{(x)} dx}{E A_{(x)}}$$

$$\delta = \int_0^{120} \frac{P_{(x)} dx}{E A_{(x)}} \quad , \quad n = 6 \text{ panels}$$

$$h = \frac{120 - 0}{6} = 20$$

$$I = \delta = \int_0^{120} \frac{10 \, dx}{30 * 10^3 \left[-\frac{1}{3600} x^2 + \frac{1}{30} x + 1 \right]}$$

$$I = \int f(x) dx$$

$$f(x) = \frac{10}{30 * 10^3 \left(-\frac{1}{3600} x^2 + \frac{1}{30} x + 1 \right)}$$

$$f(x) = \frac{10 * 10^{-3}}{30 \left(-\frac{1}{3600} x^2 + \frac{1}{30} x + 1 \right)}$$

$$f(x) = 10^{-4} * \frac{10}{3 \left(-\frac{1}{3600} x^2 + \frac{1}{30} x + 1 \right)}$$

$x:$ 0 20 40 60 80 100 120

$f(x) = y * 10^{-40}$: 3.33 2.143 1.765 1.667 1.765 2.143 3.33

Simpson's Rule :

$$S = I = \frac{10^{-4} * 20}{3} [3.33 + 4(2.143 + 1.667 + 2.147) + 2(1.765 + 1.765) + 3.33]$$

$$\therefore I = 0.02502 \text{ in}$$

Gauss – Quadrature formulas :

This formula is determined without specified pivot points that is given:

$$I = \int_A^B f(x) dx$$

The integral is approximated by a function of the form:

$$I = H_1 f(x_1) + H_2 f(x_2) + H_3 f(x_3) + \dots + H_n f(x_n)$$

If there is one (single) term $[I = H_1 f(x_1)]$;

$$\int_A^B f(x) dx = H_1 f(x_1); \text{ two unknowns } (H_1 \text{ \& } x_1)$$

So that the integration must hold for :

$$f(x) = 1 \text{ \& } x_1; \int_A^A 1 dx = H_1(1)$$

$$B - A = H_1 \dots\dots\dots(1)$$

$$\int_A^A x dx = H_1 x_1 \Rightarrow \frac{1}{2}(B^2 - A^2) = H_1 x_1 \dots\dots(2)$$

$$\text{Solve Eqs. (1) \& (2), getting; } x_1 = \frac{B - A}{2}$$

$$\therefore I = H_1 f(x_1) = (B - A) f\left(\frac{B - A}{2}\right)$$

Gauss – Quadrature formulas :

Involving two and higher term result in sets of nonlinear algebraic equations. It's for this reason that the limit of integration is normalized as follows :

$$I = \int_A^B f(x) dx, \text{ Put } x = c_o + c_1 r$$

$$x = A ; r = -1 \Rightarrow \therefore A = c_o + c_1(-1) \dots\dots\dots (1)$$

$$x = B ; r = +1 \Rightarrow \therefore B = c_o + c_1(+1) \dots\dots\dots(2)$$

$$c_o = \frac{1}{2}(A + B) \quad \text{and} \quad c_1 = \frac{1}{2}(B - A)$$

$$\therefore x = \frac{1}{2}(B + A) + \frac{1}{2}(B - A) r$$

$$dx = \frac{1}{2}(B - A) dr$$

$$I = \int_A^B f(x) dx$$

$$I = \int_{-1}^{+1} f\left[\frac{1}{2}(B + A) + \frac{1}{2}(B - A) r\right] \left[\frac{1}{2}(B - A) dr\right]$$

$$= \frac{1}{2}(B - A) \int_{-1}^{+1} f\left[\frac{1}{2}(B + A) + \frac{1}{2}(B - A) r\right] dr$$

Two terms :

$$I = H_1 f(r_1) + H_2 f(r_2)$$

$$f(r) = 1; r; r^2$$

$$\int_{-1}^{+1} 1 \, dr = H_1(1) + H_2(1) = 2 \quad \dots\dots\dots (1)$$

$$\int_{-1}^{+1} r \, dr = H_1(-a) + H_2(a) = 0 \quad \dots\dots\dots (2)$$

$$\int_{-1}^{+1} r^2 \, dr = H_1(-a)^2 + H_2(a)^2 = \frac{2}{3} \quad \dots\dots (3)$$

Solve the above Eqs., getting

$$H_1 = H_2 = 1 \quad \text{and} \quad a = \pm \frac{1}{\sqrt{3}}$$

$$\therefore \int_{-1}^1 f(r) dr = (1) f\left(\frac{-1}{\sqrt{3}}\right) + (1) f\left(\frac{1}{\sqrt{3}}\right)$$

Example (1):

$$\int_{-1}^{+1} x^2 dx \quad \text{two terms}$$

Solution :

$$f\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \left(\frac{-1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$$I = 1 * \frac{1}{3} + 1 * \frac{1}{3} = \frac{2}{3}$$

Three terms :

$$H_1 = \frac{5}{9} \quad \Leftrightarrow \quad a = -\sqrt{\frac{3}{5}}$$

$$H_2 = \frac{8}{9} \quad \Leftrightarrow \quad a = 0$$

$$H_3 = \frac{5}{9} \quad \Leftrightarrow \quad a = \sqrt{\frac{3}{5}}$$

Example (2):

$$\int_0^4 e^x dx \quad \text{three terms}$$

Solution :

$$a = -\sqrt{3/5} = -0.774 \quad H_1 = 0.555$$

$$a = 0 \quad H_2 = 0.888$$

$$a = \sqrt{3/5} = 0.774 \quad H_3 = 0.555$$

$$x = \frac{1}{2}(B + A) + \frac{1}{2}(B - A) \quad (r) \quad \underline{\underline{or (a)}}$$

$$x_1 (r = -0.774) = \frac{1}{2}(4 + 0) + \frac{1}{2}(4 - 0)(-0.774) = 0.452$$

$$x_2 (r = 0) = \frac{1}{2}(4 + 0) + \frac{1}{2}(4 - 0)(0) = 2$$

$$x_3 (r = 0.774) = \frac{1}{2}(4 + 0) + \frac{1}{2}(4 - 0)(0.774) = 3.548$$

$$f(x_1) = e^{0.452} = 1.57145$$

$$f(x_2) = e^2 = 7.38906$$

$$f(x_3) = e^{3.548} = 34.7438$$

$$I = \frac{1}{2}(4-0)[0.555 * 1.57145 + 0.888 * 7.38906 \\ + 0.555 * 34.7438]$$

$$\therefore I = 53.4463$$

Example : Evaluate $\left[\int_0^1 \left(\frac{\sin x}{x} \right) dx \right]$ *use:*

Gauss – Quadrature formula ($n = 3$)

Solution :

$$a = 0.774$$

$$H_1 = 0.555$$

$$a = 0$$

$$H_2 = 0.888$$

$$a = -0.774$$

$$H_3 = 0.555$$

$$x = \frac{1}{2}(B + A) + \frac{1}{2}(B - A)(a) \quad \underline{\underline{or (r)}}$$

$$x_1 (r = 0.774) = \frac{1}{2}(1 + 0) + \frac{1}{2}(1 - 0)(0.774) = 0.8873$$

$$x_2 (r = 0) = \frac{1}{2}(1 + 0) + \frac{1}{2}(1 - 0)(0) = 0.5$$

$$x_3 (r = -0.774) = \frac{1}{2}(1 + 0) + \frac{1}{2}(1 - 0)(-0.774) = 0.1127$$

$$f(x_1) = \frac{\sin(0.8873)}{0.8873} = 0.87385$$

$$f(x_2) = \frac{\sin(0.5)}{0.5} = 0.95885$$

$$f(x_3) = \frac{\sin(0.1127)}{0.1127} = 0.99788$$

$$I = \frac{1}{2}(1-0)[0.555 * 0.87385 + 0.888 * 0.95885 \\ + 0.555 * 0.99788]$$

$$\therefore I = 0.9451$$

Example : $\int_1^3 e^x dx$ (use $n = 4$)

Solution :

$$a = \pm 0.861136, \quad \pm 0.339981$$

$$H = 0.347854, \quad 0.652145$$

$$x = \frac{1}{2}(B + A) + \frac{1}{2}(B - A) \quad (a) \quad \underline{\underline{or (r)}}$$

$$x_1 = \frac{1}{2}(3+1) + \frac{1}{2}(3-1)(0.861136) = 2.861136$$

$$x_2 = \frac{1}{2}(3+1) + \frac{1}{2}(3-1)(-0.861136) = 1.138864$$

$$x_3 = \frac{1}{2}(3+1) + \frac{1}{2}(3-1)(0.339981) = 2.339981$$

$$x_4 = \frac{1}{2}(3+1) + \frac{1}{2}(3-1)(-0.339981) = 1.660019$$

$$I = \frac{1}{2}(B - A)[H_1 f(x_1) + H_2 f(x_2) + H_3 f(x_3) + H_4 f(x_4)]$$

$$f(x_1) = e^{2.861136} = 17.4813745$$

$$f(x_2) = e^{1.138864} = 3.123218373$$

$$f(x_3) = e^{2.339981} = 10.38103932$$

$$f(x_4) = e^{1.660019} = 5.259410772$$

$$I = \frac{1}{2}(3 - 1)[0.347854 * 17.4813745 + 0.347854 * 3.123218373 \\ + 0.652145 * 10.38103932 + 0.652145 * 5.259410772]$$

$$\therefore I = 17.36673$$