$Boundary\ \ value - problem$

$The\ method\ \ of\ \ Finite\ \ Difference:$

$$The\ derivative\ \frac{dy}{dx}\ is\ \ the\ ratio\ \ of\ \ very\ \ small$$

$(dy)\ \ over\ \ very\ \ small\ \ (dx).$

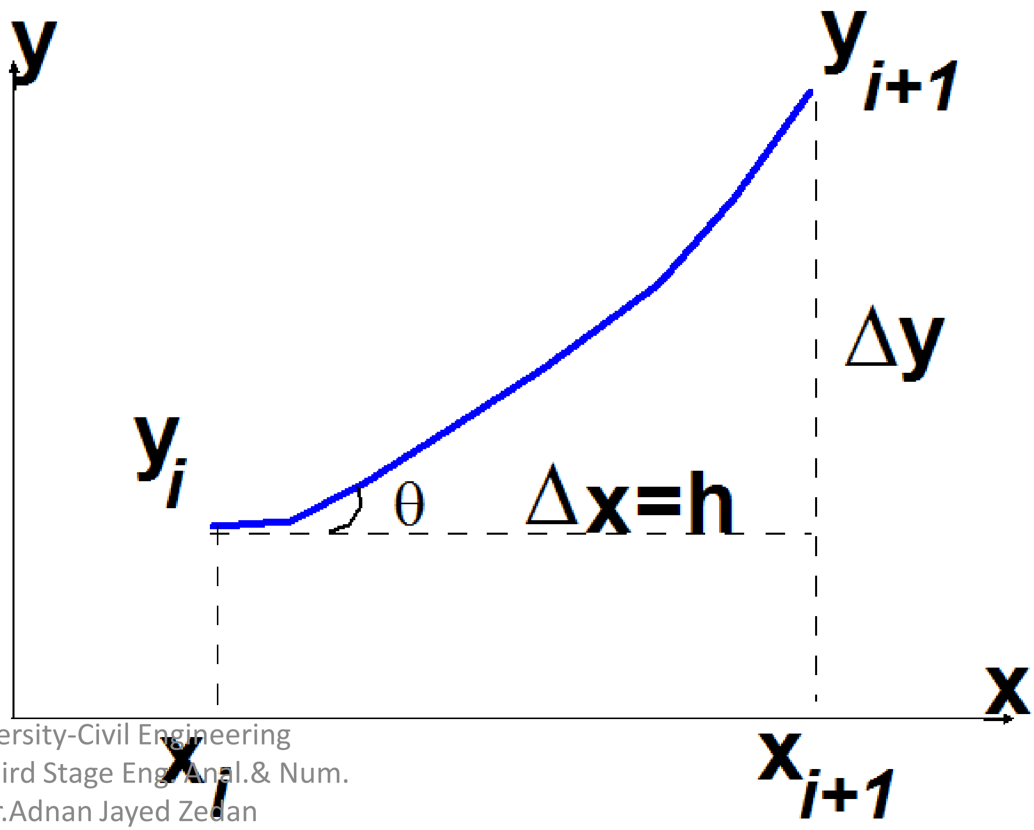
$In\ \ finite\ \ difference\ \ technique,\ the\ \ derivative\ \frac{dy}{dx}\ \ is$

$given\ \ as\ \ \frac{\Delta y}{\Delta x}.$

$$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

$$(x_{i+1} - x_i = h), \ \ where;$$

$$(h \ is \ \ the \ \ step \ \ size)$$

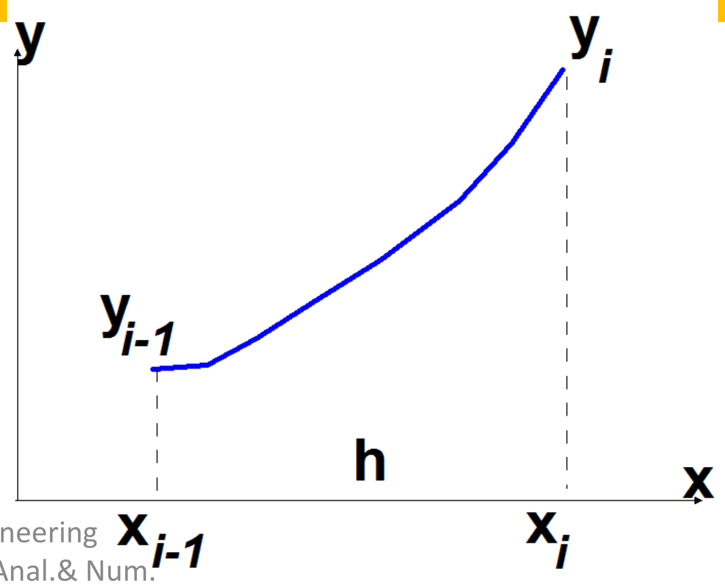


$$\underline{1-Forward-method:}$$

$$\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=\frac{y_{i+1}-y_i}{h} \qquad (forward \;\; difference)$$

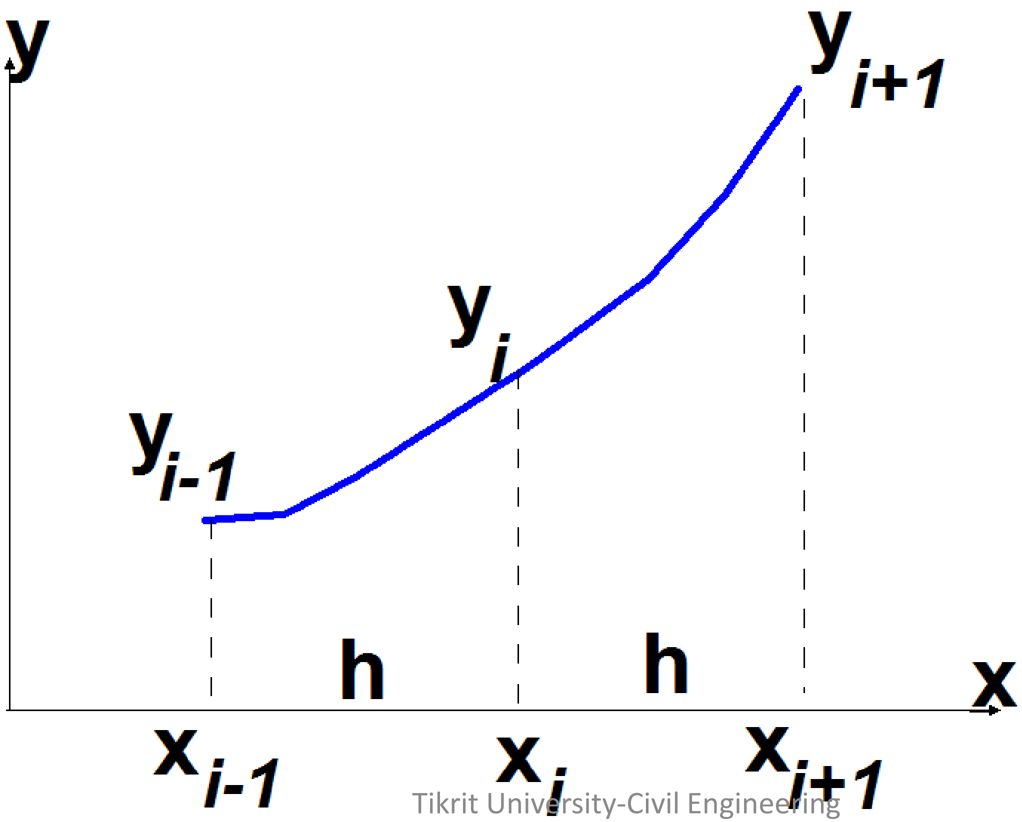
$$\underline{2-Backward-method:}$$

$$\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=\frac{y_i-y_{i-1}}{x_i-x_{i-1}}=\frac{y_i-y_{i-1}}{h} \qquad (backward \;\; difference)$$



$3-Central\ \ difference$

$$\frac{dy}{dx}=\frac{\Delta y}{\Delta x}=\frac{y_{i+1}-y_{i-1}}{x_{i+1}-x_{i-1}}=\frac{y_{i+1}-y_{i-1}}{2h} \qquad (central\ \ difference)$$



$\underline{Second\ \ derivative:}$ $\qquad \dfrac{\Delta^2 y}{\Delta x^2}$

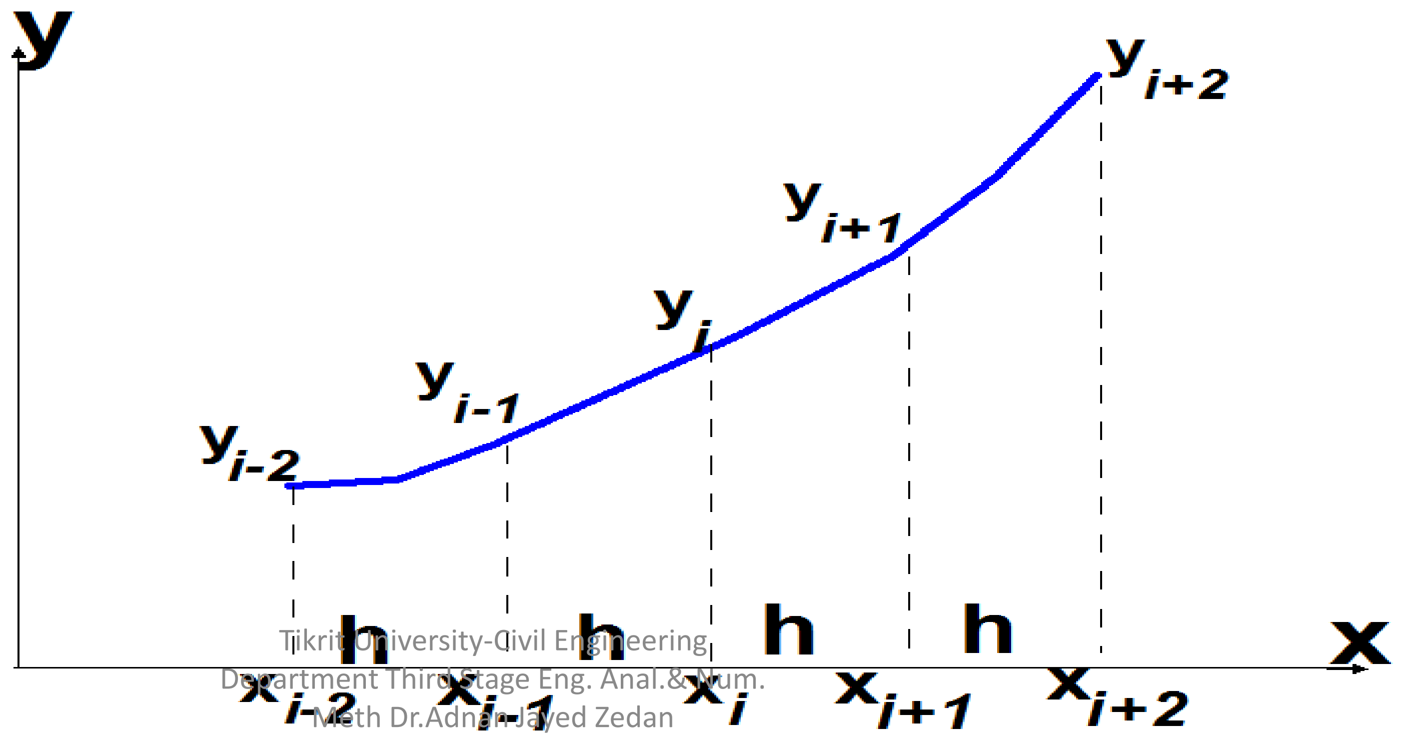
$$\frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{h}$$

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{\left(\dfrac{y_{i+1} - y_i}{h}\right) - \left(\dfrac{y_i - y_{i-1}}{h}\right)}{h} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Fourth derivative:

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$\frac{\Delta^4 y}{\Delta x^4} = \frac{\Delta^2}{\Delta x^2}\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right)$$



$$\frac{\Delta^4 y}{\Delta x^4} = \frac{y''_{i+1} - 2y''_i + y''_{i-1}}{h^2}$$

$$= \frac{\left(\dfrac{y_{i+2} - 2y_{i+1} + y_i}{h^2}\right) - 2\left(\dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(\dfrac{y_i - 2y_{i-1} + y_{i-2}}{h^2}\right)}{h^2}$$

$$\therefore \frac{\Delta^4 y}{\Delta x^4} = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4}$$

In Brief :

$$\frac{\Delta y}{\Delta x}=\frac{1}{h}\begin{bmatrix}-1 & +1\\ y_i & y_{i+1}\end{bmatrix}$$ *forward difference*

$$\frac{\Delta y}{\Delta x}=\frac{1}{h}\begin{bmatrix}-1 & +1\\ y_{i-1} & y_i\end{bmatrix}$$ *backward difference*

$$\frac{\Delta y}{\Delta x}=\frac{1}{2h}\begin{bmatrix}-1 & +1\\ y_{i-1} & y_{i+1}\end{bmatrix}$$ *central difference*

$$\frac{\Delta^2 y}{\Delta x^2}=\frac{1}{h^2}\begin{bmatrix}1 & -2 & 1\\ i-1 & i & i+1\end{bmatrix}$$

$$\frac{\Delta^4 y}{\Delta x^4}=\frac{1}{h^4}\begin{bmatrix}1 & -4 & 6 & -4 & 1\\ i-2 & i-1 & i & i+1 & i+2\end{bmatrix}$$

$$\left(\sum Coefficien\ ts=0\right)$$

Application to the solution of differential equations:

The condition required depends on the order of the D.E.:

i) 2^{nd} order D.E. requires two conditions:

These are given at the beginning and at the end of the domain.

ii) 4^{th} order D.E. requires four conditions:

Each two are given at the beginning and at the end of the domain.

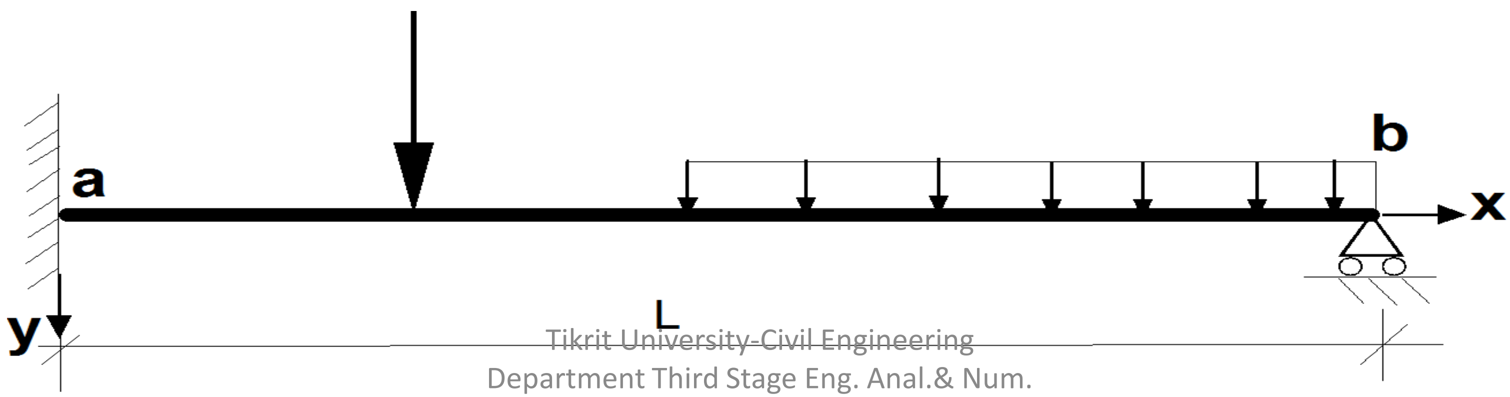
$$\frac{d^4 y}{dx^4} = \frac{q}{EI}$$

$\underline{B.conditions}:$

$$At \quad x = 0 \Rightarrow y = 0 \ and \ \frac{dy}{dx} = 0$$

$$At \quad x = L \Rightarrow y = 0 \ and \ M = 0$$

$$\frac{d^2 y}{dx^2} = \frac{M}{EI} = 0$$



$$\underline{B.conditions}:$$

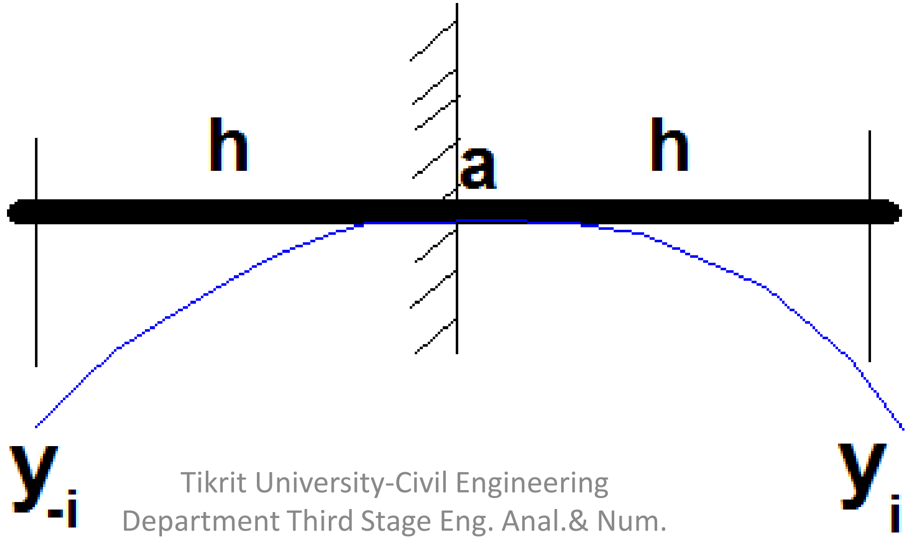
$$At \quad x=0 \Rightarrow y=0 \ and \ \frac{dy}{dx}=0$$

$B.\ conditions\ \ of\ \ \ fixed\ \ end:$

$$Slope\ \ at\ \ a = 0$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{2h}\left[-y_{-i} \quad + y_i\right] = 0$$

$$\therefore y_{-i} = y_i$$

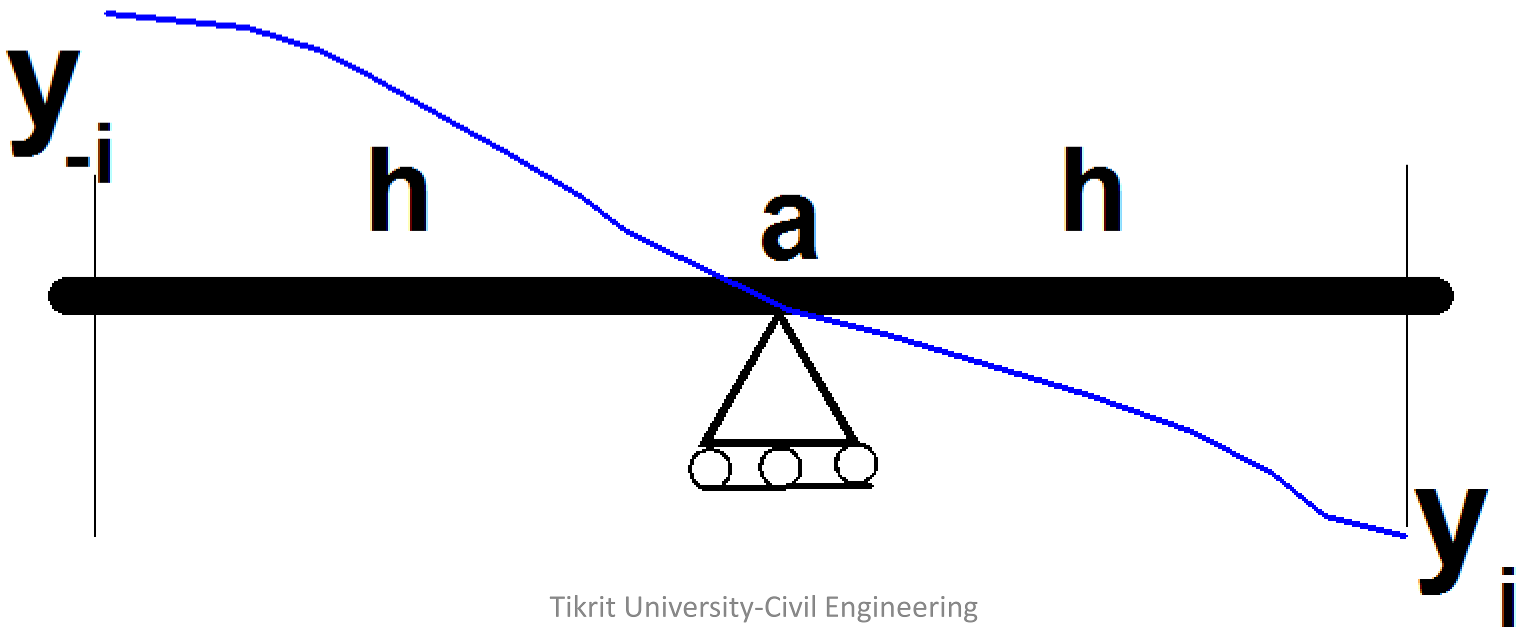


$$B.\ conditions\ \ of\ \ simply\ \sup ported\ \ beam:$$

$$Moment\ at\ \ a=0$$

$$M=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left[y_{-i}\quad -2*0\quad y_i\right]=0$$

$$\therefore y_{-i}=-y_i$$



$Applicatio\,ns:$

$Example\,(1):$

$$Solve\quad \frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=x$$

$$y(0)=0;\qquad y(1)=6.5;\qquad h=0.2$$

$$\underline{\underline{Solution}}:$$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x$$

$$\frac{1}{h^2}[y_{i-1} - 2y_i + y_{i+1}] - 3\frac{1}{h}[y_i - y_{i-1}] + 2y_i = x_i$$

$$For\ \ y_1=?\quad i=1$$

$$\frac{1}{(0.2)^2}[0-2y_1+y_2]-3\frac{1}{0.2}[y_1-0]+2y_1=0.2\ ......(1)$$

$$For\ \ y_2=?\quad i=2$$

$$\frac{1}{(0.2)^2}[y_1-2y_2+y_3]-3\frac{1}{0.2}[y_2-y_1]+2y_2=0.4\ ......(2)$$

$$For\ \ y_3=?\quad i=3$$

$$\frac{1}{(0.2)^2}[y_2-2y_3+y_4]-3\frac{1}{0.2}[y_3-y_2]+2y_3=0.6\ ......(3)$$

$$For\ \ y_4=?\quad i=4$$

$$\frac{1}{(0.2)^2}[y_3-2y_4+6.5]-3\frac{1}{0.2}[y_4-y_3]+2y_4=0.8\ ......(4)$$

$\underline{\underline{H.W.:}}$

$Solve \ \ for \ \ y_1, \ y_2, \ y_3 \ \ and \ \ y_4 \ using:$

$1 - Gauss - elimination$

$2 - Cramer's \ Rule$

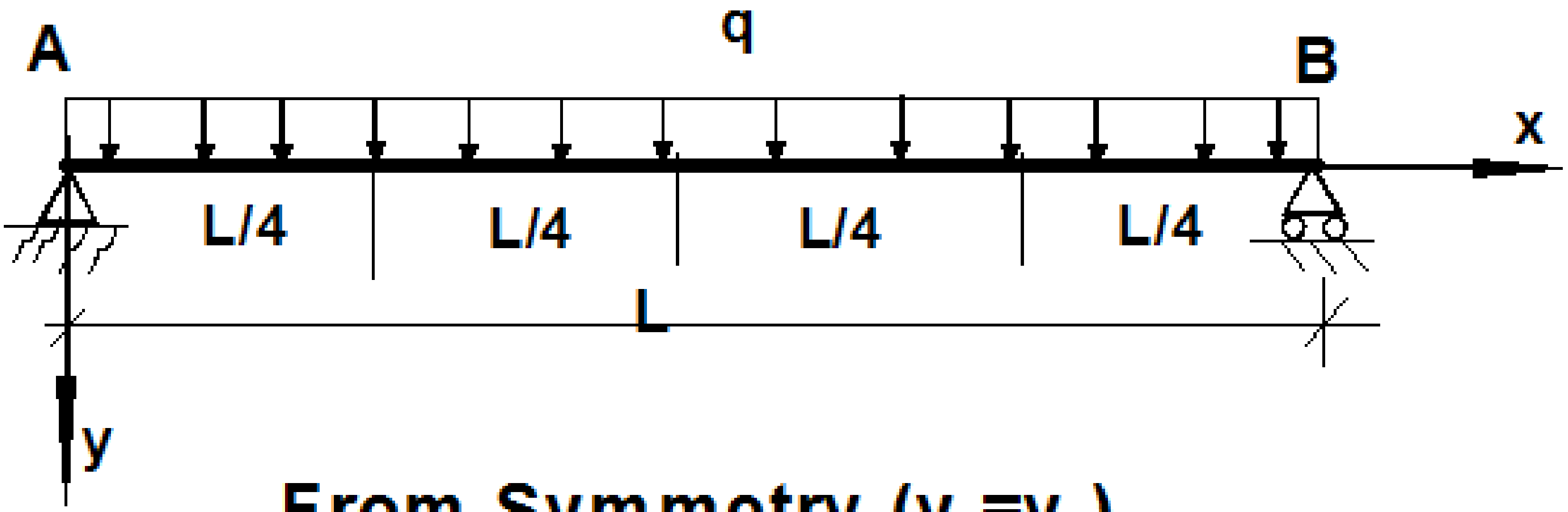
$3 - Matrix \ \ inverse$

$4 - Iteration$

Example (2):

A simply supported beam of length (L) and constant flexural rigidity [EI], a uniform load it is carried of (q) per unit length. Find the deflection, use step size ($h=L/4$).

Solution:



From Symmetry ($y_1 = y_3$)

$$Use\ \frac{d^4y}{dx^4}=\frac{q}{EI}$$

$$\frac{1}{h^4}[y_{i-2}-4y_{i-1}+6y_i-4y_{i+1}+y_{i+2}]=\frac{q_i}{EI}$$

$$For \quad y_1 = ? \quad ; i = 1$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[-y_1 - 4(0) + 6y_1 - 4y_2 + y_1] = \frac{q}{EI}$$

$$6y_1 - 4y_2 = \frac{q}{EI}\left(\frac{L}{4}\right)^4 \ldots\ldots\ldots\ldots(1)$$

$$For\ \ y_2 = ?\ \ ;\ i = 2$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[0 - 4y_1 + 6y_2 - 4y_1 + 0] = \frac{q}{EI}$$

$$-8y_1 + 6y_2 = \frac{q}{EI}\left(\frac{L}{4}\right)^4 \ldots\ldots\ldots\ldots(2)$$

$$Solve\ \ Eqs.\,1\&2\ \ getting\ :$$

$$y_1 = \frac{5}{512}\frac{qL^4}{EI};\quad y_1 = \frac{7}{512}\frac{qL^4}{EI} = 0.013672\frac{qL^4}{EI}$$

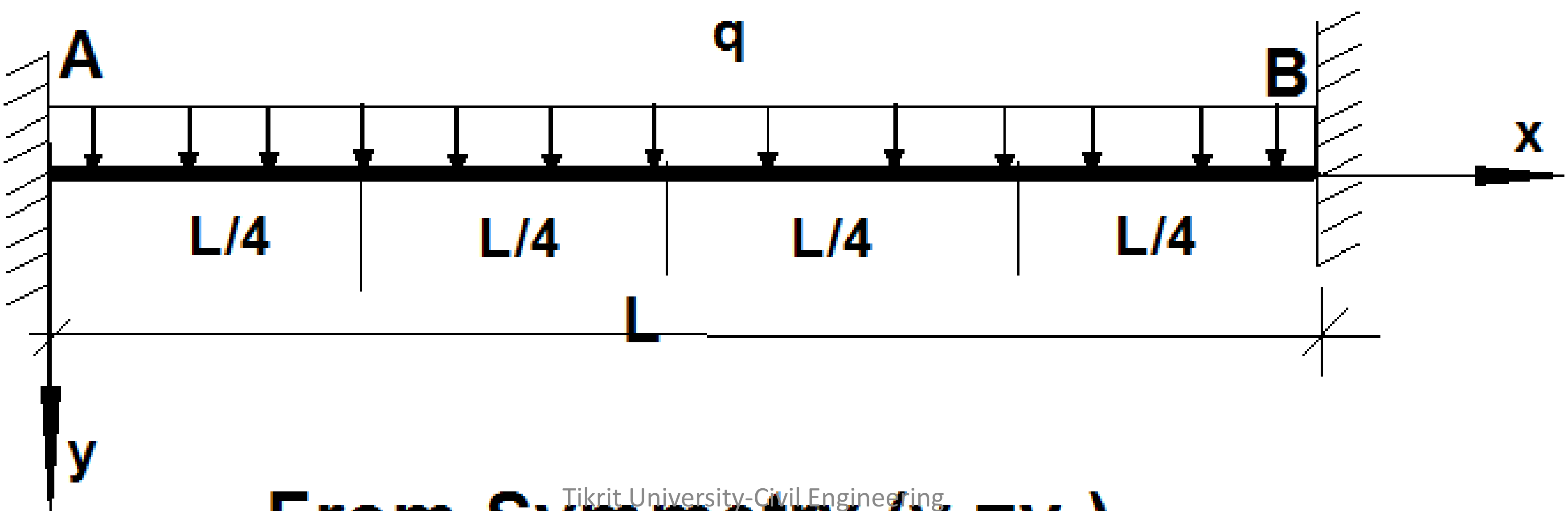
$$y_2\ \ exact = \frac{5qL^4}{384EI} = 0.013021\frac{qL^4}{EI}$$

$$Error \approx 5\%$$

Example (3):

If (A) and (B) are fixed; find the deflections and the fixed moments.

Solution:



From Symmetry ($y_1 = y_3$)

$$Use\ \frac{d^4 y}{dx^4} = \frac{q}{EI}$$

$$\frac{1}{h^4}[y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}] = \frac{q}{EI}$$

$$For \quad y_1 = ? \quad ; i = 1$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[y_1 - 4(0) + 6y_1 - 4y_2 + y_1] = \frac{q}{EI}$$

$$8y_1 - 4y_2 = \frac{q}{EI}\left(\frac{L}{4}\right)^4 = A \ldots\ldots\ldots\ldots(1)$$

$$For\ \ y_2 = ?\ \ ;\, i = 2$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[0 - 4y_1 + 6y_2 - 4y_1 + 0] = \frac{q}{EI}$$

$$-8y_1 + 6y_2 = A \quad \text{.............}(2)$$

$$Solve\ \ Eqs.\ 1\&2\ \ getting:$$

$$y_1 = \frac{5}{8}A \qquad ; \quad y_2 = A$$

$$y_1 = \frac{5}{8}\frac{q}{EI}\left(\frac{L}{4}\right)^4 = \frac{5qL^4}{2048EI}$$

$$y_2 = \frac{qL^4}{256EI}$$

$$M_A = ???$$

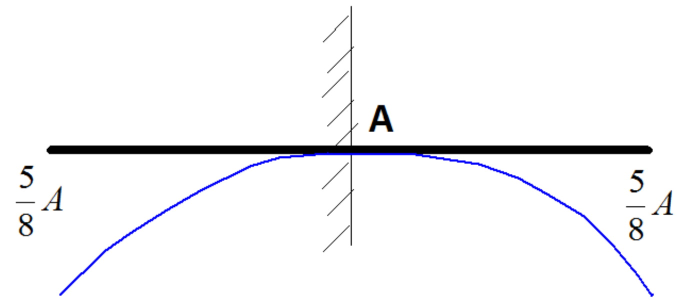
$$\frac{d^2y}{dx^2} = \frac{M}{EI} \Rightarrow M = EI\frac{d^2y}{dx^2}$$

$$M = EI\frac{1}{\left(\frac{L}{4}\right)^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$$

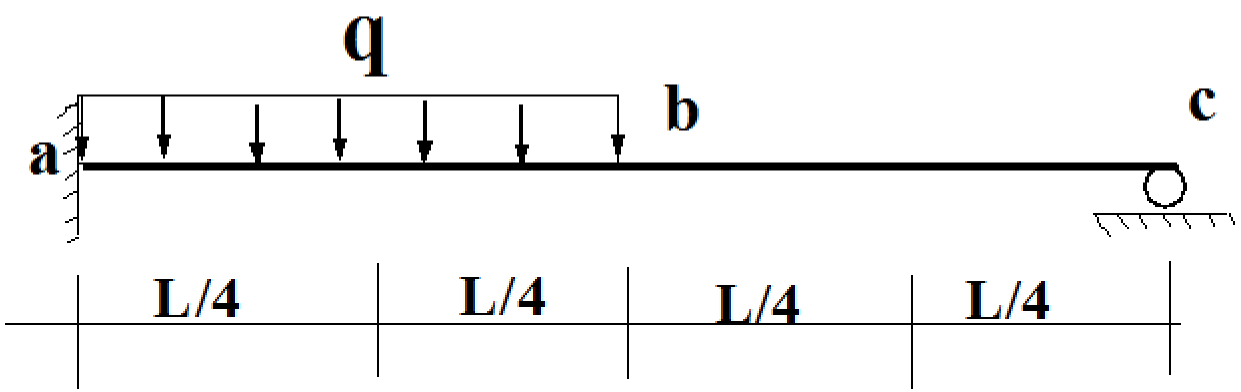
$$= \frac{16EI}{L^2}\left[\frac{5qL^4}{2048EI} - 0 + \frac{5qL^4}{2048EI}\right]$$

$$\therefore M = \frac{160}{2048}qL^2 = 0.078125qL^2$$

$$Exact \quad M = \frac{qL^2}{12} = 0.0833qL^2$$



$$\underline{\underline{Ex}}: \quad h = \Delta x = \frac{L}{4}; \quad EI \ \text{constant}$$



Solution

$$Use\ \frac{d^4y}{dx^4}=\frac{q}{EI}$$

$$\frac{1}{h^4}[y_{i-2}-4y_{i-1}+6y_i-4y_{i+1}+y_{i+2}]=\frac{q}{EI}$$

$$For\ \ y_1 = ?\ \ ;\, i = 1$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}\left[y_1 - 4(0) + 6y_1 - 4y_2 + y_3\right] = \frac{q}{EI}$$

$$7y_1 - 4y_2 + y_3 = \frac{q}{EI}\left(\frac{L}{4}\right)^4 = A \ldots\ldots\ldots\ldots (1)$$

$$For \ \ y_2 = ? \ \ ; i = 2$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[0 - 4y_1 + 6y_2 - 4y_3 + 0] = \frac{1}{2}\frac{q}{EI}$$

$$-4y_1 + 6y_2 - 4y_3 = \frac{1}{2}A \quad ....(2)$$

$$For\ \ y_3 = ?\ \ ; i = 3$$

$$\frac{1}{\left(\frac{L}{4}\right)^4}[y_1 - 4y_2 + 6y_3 - 4(0) - y_3] = 0$$

$$y_1 - 4y_2 + 5y_3 = 0 \quad \ldots\ldots\ldots\ldots.(3)$$

$$Solve\ \ Eqs.\ (1), (2)\ \&\ (3),\ getting\ \ y_1,\ y_2\ \&\ y_3$$

$\underline{Solution\ \ of\ \ \ P.D.E.}:$

$\underline{\underline{Solution\ \ of\ \ (2D)\ Steady\ \ Head\ \ Flow}}:$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

$$T(x,y)$$

$\underline{Solution\ \ of\ \ \ P.D.E.}:$

$\underline{\underline{Solution\ \ of\ \ (2D)\ Steady\ \ Head\ \ Flow}}:$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

$$T(x,y)$$

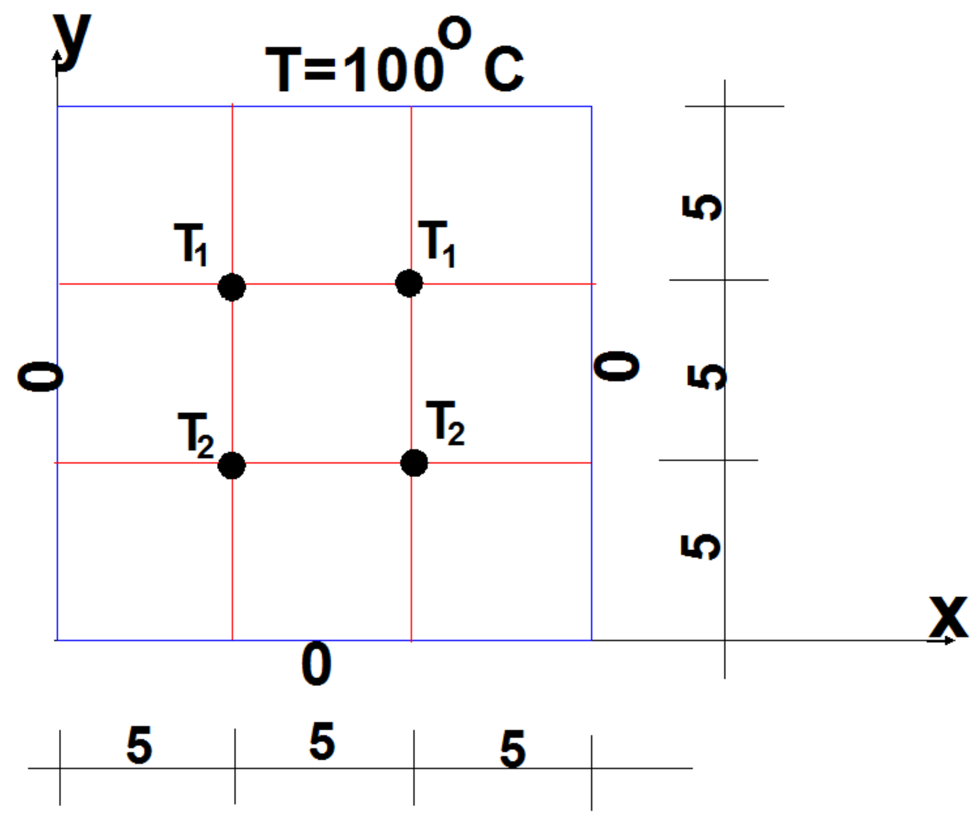
$$\underline{\underline{Boundary\ \ Conditions:}}$$

$$T(x,15) = 100$$

$$T(x,0) = 0$$

$$T(0,y) = 0$$

$$T(15,y) = 0$$



Grid mesh (3x3)

$$In\ \ general:$$

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{d^2 y}{dx^2} = \frac{1}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

$$For\ \ T_1 = ? \quad i = 1$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{1}{5^2}[0 - 2T_1 + T_1] + \frac{1}{5^2}[T_2 - 2T_1 + 100] = 0 \ .....(1)$$

$$For\ \ T_2 = ? \quad i = 2$$

$$\frac{1}{5^2}[0 - 2T_2 + T_2] + \frac{1}{5^2}[0 - 2T_2 + T_1] = 0 \ .....(2)$$

$$Solve\ \ Eqs.\ (1)\ \&\ (2),\ getting\ :$$

$$T_1 = 37.5^o C, and\ \ T_2 = 12.5^o C$$