Fourier Series and Integral :

$$f(t)=\frac{a_o}{2}+a_1\cos\frac{\pi t}{p}+a_2\cos\frac{2\pi t}{p}+a_3\cos\frac{3\pi t}{p}+......+a_n\cos\frac{n\pi t}{p}$$

$$b_1\sin\frac{\pi t}{p}+b_2\sin\frac{2\pi t}{p}+b_3\sin\frac{3\pi t}{p}+......+b_n\sin\frac{n\pi t}{p}$$

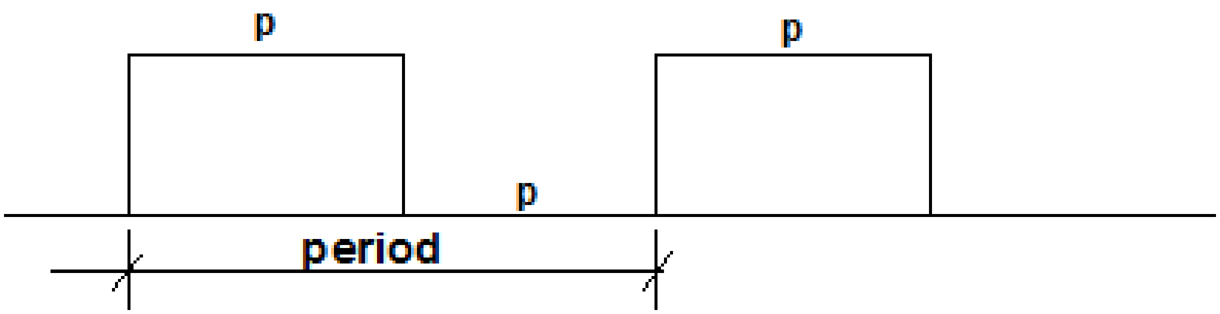
$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{p}+b_n\sin\frac{n\pi t}{p}\right)$$

$$a_o = \frac{1}{p} \int_{d}^{d+2p} f(t)\, dt$$

$$a_n = \frac{1}{p} \int_{d}^{d+2p} f(t) \cos\frac{n\pi t}{p} dt$$

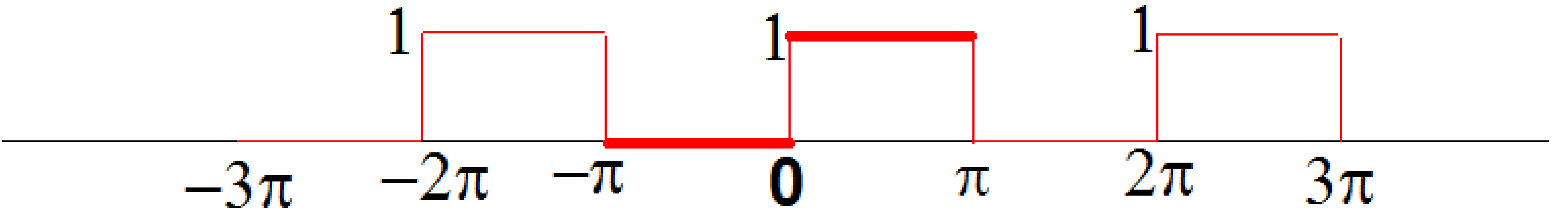
$$b_n = \frac{1}{p} \int_{d}^{d+2p} f(t) \sin\frac{n\pi t}{p} dt$$

$$2p = period$$



Example :

$$f(x)=\begin{cases}0 & -\pi<x<0\\ 1 & 0<x<\pi\end{cases}$$



Solution :

$$p = \pi$$

$$a_o = \frac{1}{p}\int_{d}^{d+2p} f(t)\; dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} 1\, dx\right] = \frac{1}{\pi} * \pi = 1$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\, \cos\frac{n\pi t}{p}\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, \cos nx\, dx + \int_{0}^{\pi} 1 \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\, dx = \frac{1}{\pi}\frac{1}{n}\sin nx\Big|_{0}^{\pi} = 0 \qquad n \neq 0$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\,\sin\frac{n\pi t}{p}\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,\sin nx\,dx + \int_{0}^{\pi} 1\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx = \frac{1}{\pi}\frac{1}{n}(-\cos nx)\Big|_{0}^{\pi}$$

$$= \frac{1}{n\pi}(-\cos n\pi - (-\cos 0)) = \frac{1}{n\pi}(1-\cos n\pi)$$

$$= \frac{1}{n\pi}(1-(-1)^n) = \begin{cases} 0 & n = even \\ \dfrac{2}{n\pi} & n = odd \end{cases}$$

$$\therefore b_n = \frac{2}{n\pi} \qquad for\ \ n = odd$$

$$a_o = 1$$

$$a_n = 0$$

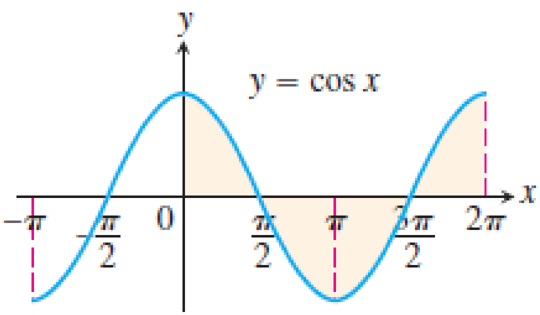
$$b_n = \frac{2}{n\pi} \qquad for \;\; n = odd$$

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right]$$

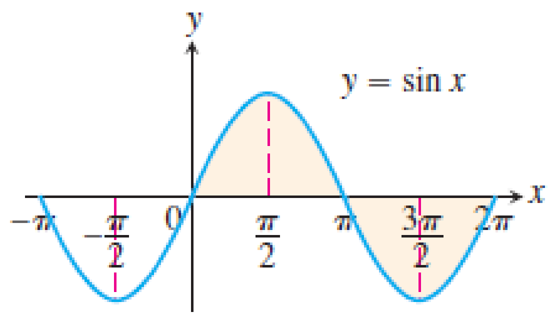
$$= \frac{1}{2} + \sum_{n=1,3,5}^{n} \frac{2}{n\pi}\sin\frac{n\pi x}{\pi}$$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5}^{n} \frac{1}{n}\sin nx$$

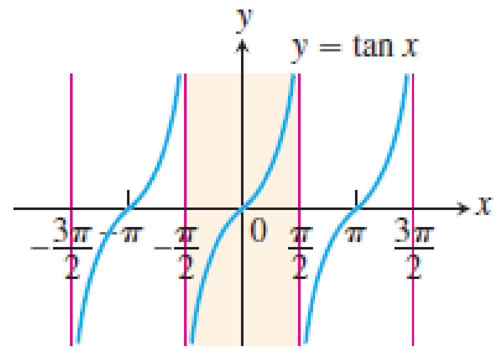
$$\therefore\; f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \ldots.\right]$$



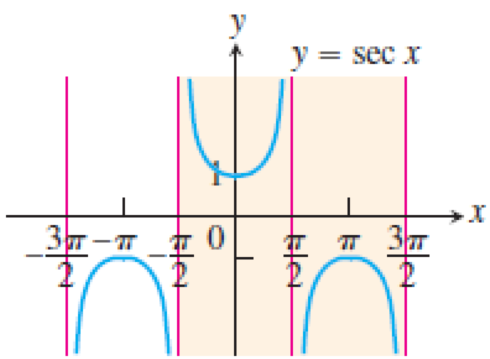
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(a)



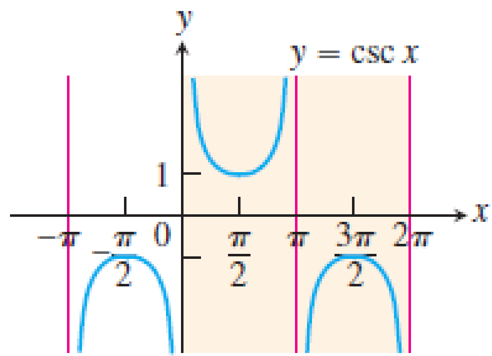
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(b)



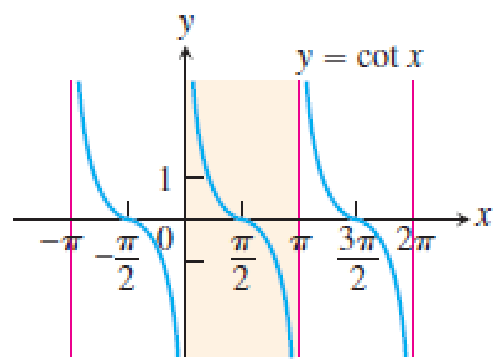
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π
(c)



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π
(d)

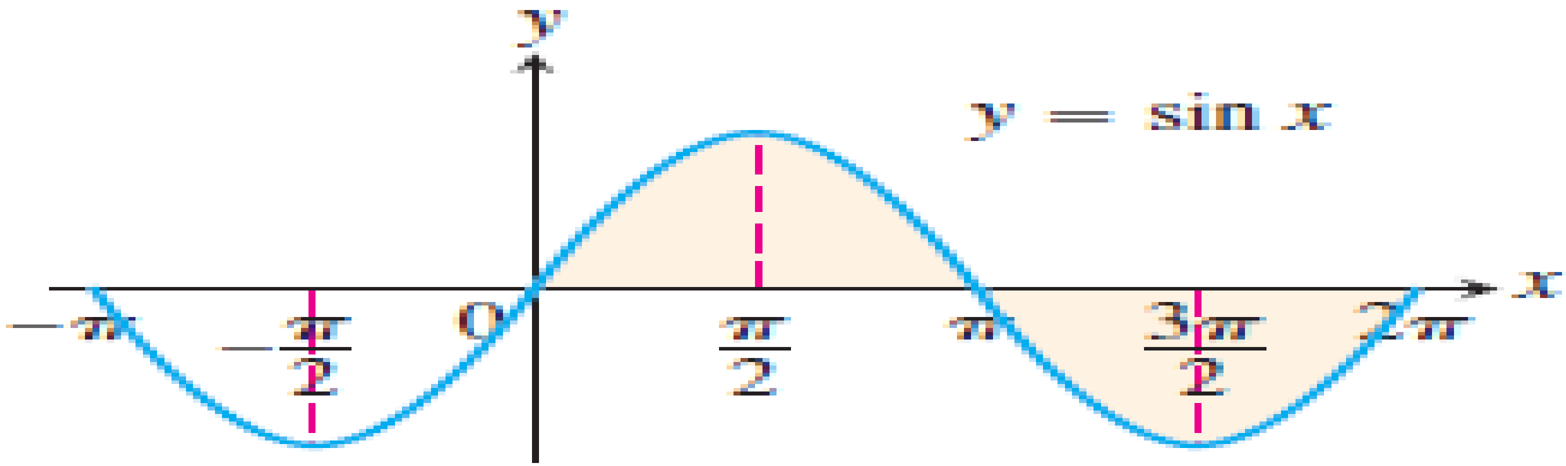


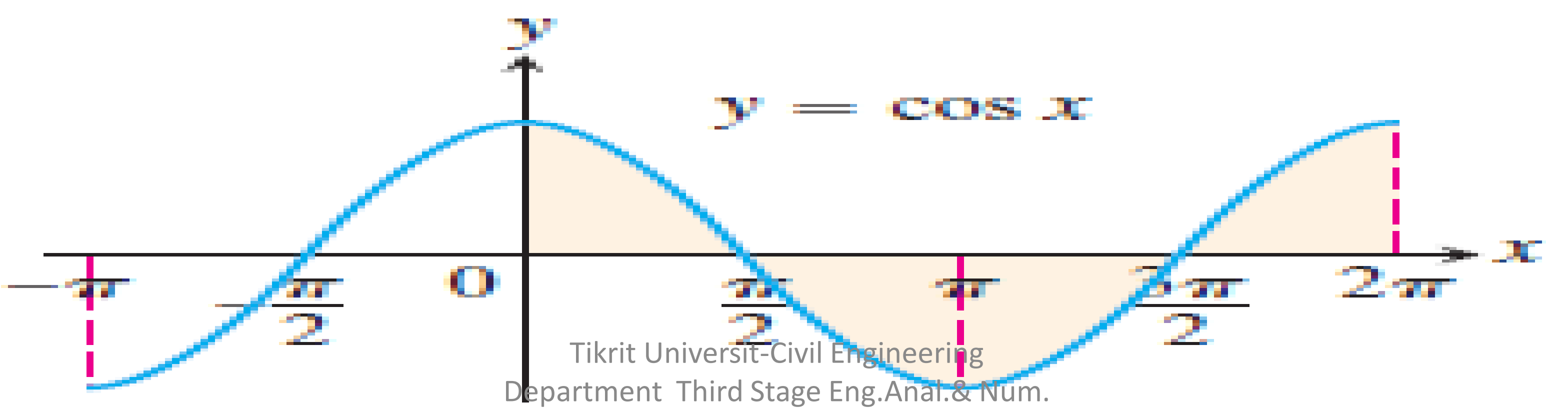
Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π
(e)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π
(f)

To obtain formula the coefficients a_n and b_n, we need the following integrals, where m and n are integers.





$$1-\int_{d}^{d+2p}\cos\frac{n\pi t}{p}\,dt=0 \qquad n\neq 0$$

$$2-\int_{d}^{d+2p}\sin\frac{n\pi t}{p}\,dt=0$$

$$3-\int_{d}^{d+2p}\cos\frac{m\pi t}{p}\sin\frac{n\pi t}{p}\,dt=0 \qquad m\neq n$$

$$4-\int_{d}^{d+2p}\cos^{2}\frac{m\pi t}{p}\,dt=p \qquad n\neq 0$$

$$5-\int_{d}^{d+2p}\cos\frac{m\pi t}{p}\sin\frac{n\pi t}{p}\,dt=0 \qquad m=n$$

$$6-\int_{d}^{d+2p}\sin\frac{m\pi t}{p}\sin\frac{n\pi t}{p}\,dt=0 \qquad m\neq n$$

$$7-\int_{d}^{d+2p}\sin^{2}\frac{n\pi t}{p}\,dt=p \qquad n\neq 0$$

Limits :

$$1-\lim_{n\to\infty}\frac{\ln n}{n}=0$$

$$2-\lim_{n\to\infty}\sqrt[n]{n}=1$$

$$3-\lim_{n\to\infty}x^{\frac{1}{n}}=1 \qquad (x>0)$$

$$4-\lim_{n\to\infty}x^{n}=0 \qquad (|x|<1)$$

$$5-\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}=e^{x}$$

$$6-\lim_{n\to\infty}\frac{x^{n}}{n!}=0$$

$\underline{\underline{Integrals:}}$ $(m \text{ and } n \text{ are integers})$

$$1- \int_0^{2\pi} \sin nx\, dx = \int_0^{2\pi} \cos nx\, dx = 0$$

$$2- \int_0^{2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_0^{2\pi} [\cos(m-n)x - \cos(m+n)x]\, dx = 0$$

$$3- \int_0^{2\pi} \sin^2 nx\, dx = \frac{1}{2}\int_0^{2\pi} (1-\cos 2nx)\, dx = \pi$$

$$4- \int_0^{2\pi} \cos mx \cos nx\, dx = 0$$

$$5- \int_0^{2\pi} \cos mx \sin nx\, dx = 0$$

$$6- \int_0^{2\pi} \cos^2 nx\, dx = 0$$

$$7- \int_0^{2\pi} \sin nx \cos nx\, dx = \frac{1}{2}\int_0^{2\pi} \sin 2nx\, dx = 0$$

$$8-\int \sin ax \ \cos bx \ dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + c \qquad a^2 \neq b^2$$

$$9-\int \sin ax \ \sin bx \ dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + c \qquad a^2 \neq b^2$$

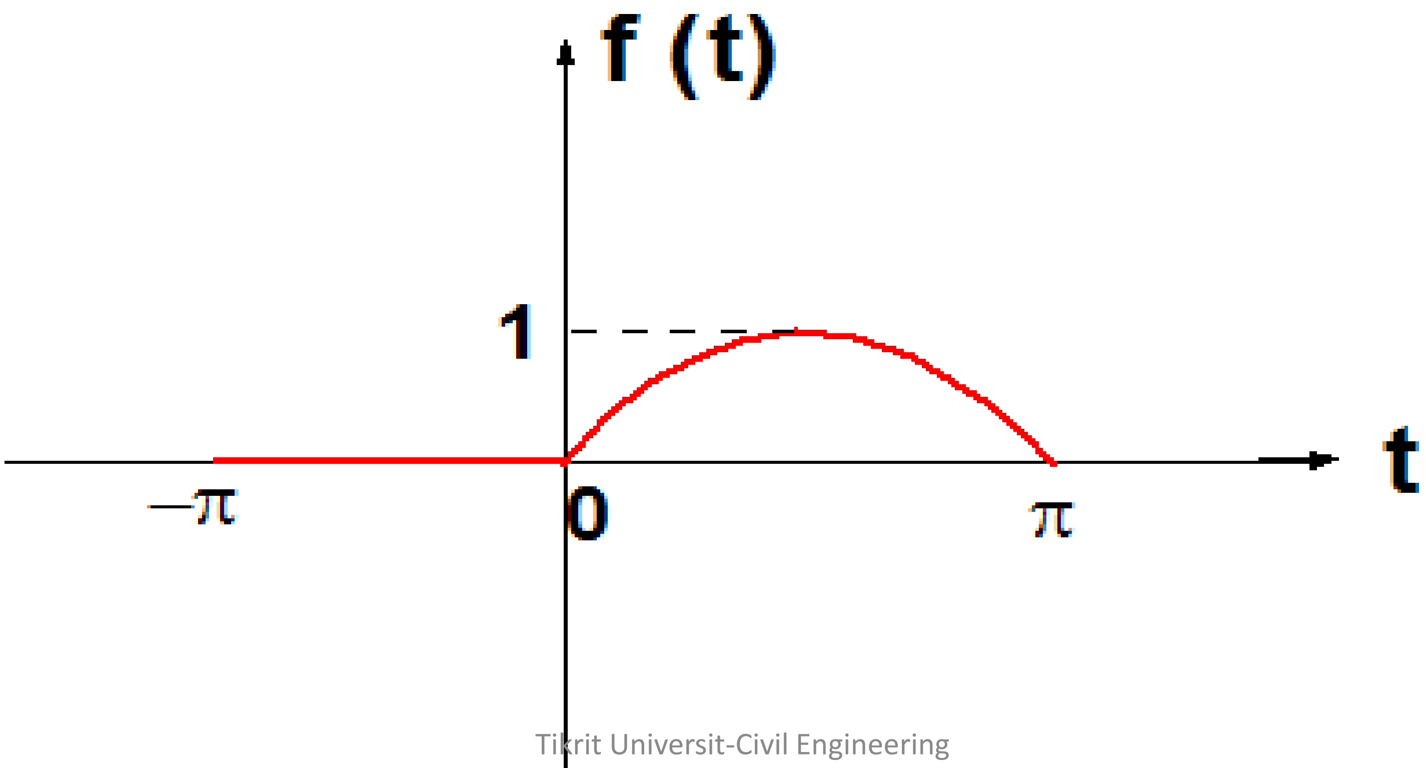
$$10-\int \cos ax \ \cos bx \ dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + c \qquad a^2 \neq b^2$$

$$11-\int \sin^2 ax \ dx = \frac{x}{2} - \frac{\sin ax}{4a} + c$$

$$12-\int \cos^2 ax \ dx = \frac{x}{2} + \frac{\sin ax}{4a} + c$$

$\underline{Example:}$ *What is the Fourier expansion of the periodic function whose definition in one period is:*

$$f(x)=\begin{cases} 0 & -\pi<t<0 \\ \sin t & 0<t<\pi \end{cases}$$



Solution :

$$a_o = \frac{1}{p}\int_{d}^{d+2p} f(t)\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dt + \int_{0}^{\pi} \sin t\,dt\right] = \frac{1}{\pi}\left(-\cos t\Big|_{0}^{\pi}\right.$$

$$= \frac{1}{\pi}[-(\cos\pi - \cos 0)] = \frac{1}{\pi}[-(-1-1)] = \frac{2}{\pi}$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\frac{n\pi t}{p}\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos\frac{n\pi t}{\pi}\,dt + \int_{0}^{\pi}\sin t\cos\frac{n\pi t}{\pi}\,dt\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\sin t\cos nt\,dt$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1-n)t}{2(1-n)} - \frac{\cos(1+n)t}{2(1+n)}\right]_0^{\pi}$$

$$= -\frac{1}{2\pi}\left[\left(\frac{\cos(\pi-n\pi)}{1-n} + \frac{\cos(\pi+n\pi)}{1+n}\right) - \left(\frac{1}{1-n} + \frac{1}{1+n}\right)\right]$$

$$a_n = -\frac{1}{2\pi}\left[\left(\frac{\cos(\pi - n\pi)}{1-n} + \frac{\cos(\pi + n\pi)}{1+n}\right) - \left(\frac{1}{1-n} + \frac{1}{1+n}\right)\right]$$

$$= -\frac{1}{2\pi}\left(\frac{-\cos n\pi}{1-n} + \frac{-\cos n\pi}{1+n} - \frac{2}{1-n^2}\right)$$

$$= -\frac{1}{2\pi}\left(\frac{-2\cos n\pi}{1-n^2} - \frac{2}{1-n^2}\right)$$

$$= \frac{1+\cos n\pi}{\pi(1-n^2)} \qquad n \neq 1 \quad (a_n = 0 \quad n = odd)$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin t \ \cos t \ dt = \left.\frac{\sin^2 t}{2\pi}\right|_0^{\pi} = 0$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\sin\frac{n\pi t}{p}\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin\frac{n\pi t}{\pi}\,dt + \int_{0}^{\pi}\sin t\sin\frac{n\pi t}{\pi}\,dt\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\sin t\sin nt\,dt$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-n)t}{2(1-n)} - \frac{\sin(1+n)t}{2(1+n)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin(\pi - n\pi)}{1-n} + \frac{\sin(\pi + n\pi)}{1+n}\right]$$

$$b_n = \frac{1}{2\pi}\left[\frac{\sin(\pi - n\pi)}{1-n} + \frac{\sin(\pi + n\pi)}{1+n}\right]$$

$$= \frac{1}{2\pi}\left(\frac{\sin n\pi}{1-n} - \frac{-\sin n\pi}{1+n}\right)$$

$$= \frac{\sin n\pi}{\pi(1-n^2)} = 0 \qquad n \neq 1$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 t\, dt \quad = \frac{1}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2t}{2}\right)dt$$

$$= \frac{1}{\pi}\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi} = \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2}$$

$$a_o = \frac{2}{\pi}$$

$$a_n = \begin{cases} 0 & n = odd \\ \dfrac{1+\cos n\pi}{\pi(1-n^2)} & n = even \end{cases} \quad (n \neq 1)$$

$$a_1 = 0$$

$$b_n = 0 \qquad (n \neq 1)$$

$$b_1 = \frac{1}{2}$$

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right]$$

$$\therefore\ f(t) = \frac{1}{\pi} + \frac{\sin t}{2} - \frac{2}{\pi} \left[\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \ldots.. \right]$$

$\underline{Note}:$

$$\cos(\pi - n\pi) = \cos\pi\cos n\pi + \sin\pi\sin n\pi = -\cos n\pi$$

$$\cos(\pi + n\pi) = \cos\pi\cos n\pi - \sin\pi\sin n\pi = -\cos n\pi$$

$$Numerical\ \ Series\ \ can\ be\ obtained\ \ from\ \ Fourier\ \ series,$$

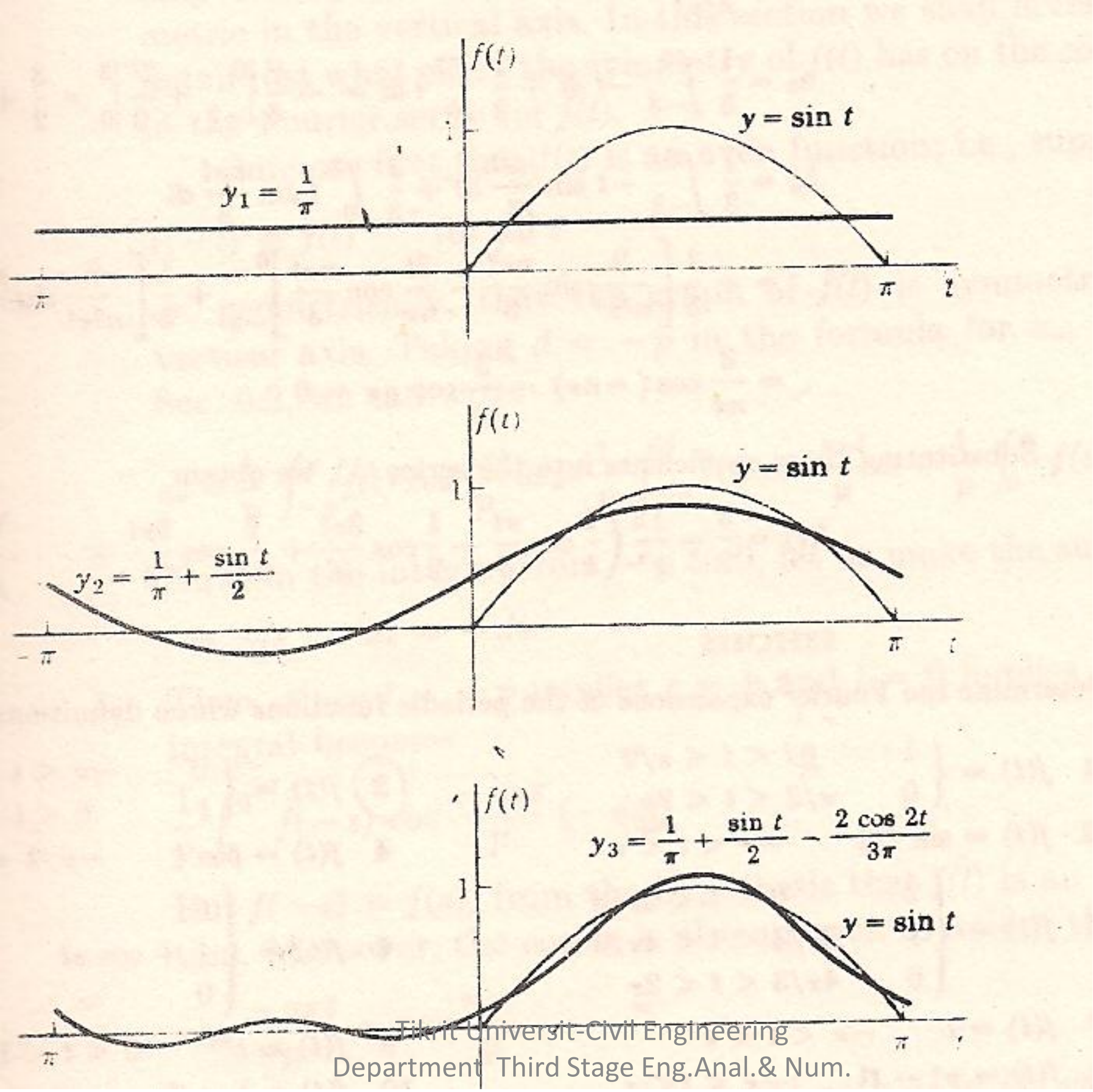
$$for\ \ example: If\ \ t=\frac{\pi}{2} in\ the\ \ previous\ \ example:$$

$$\sin\frac{\pi}{2}=f(\frac{\pi}{2})$$

$$1\ =\frac{1}{\pi}+\frac{1}{2}-\frac{2}{\pi}(-\frac{1}{3}+\frac{1}{15}-\frac{1}{35}+\frac{1}{63}-\frac{1}{99}+\frac{1}{143}........)$$

$$(1-\frac{1}{\pi}-\frac{1}{2})*\frac{\pi}{2}=\frac{1}{1*3}-\frac{1}{3*5}+\frac{1}{5*7}-\frac{1}{7*9}+\frac{1}{9*11}-\frac{1}{11*13}+.......$$

$$f(t)=\frac{1}{\pi}+\frac{\sin t}{2}-\frac{2}{\pi}\left[\frac{\cos 2t}{3}+\frac{\cos 4t}{15}+\frac{\cos 6t}{35}+.....\right]$$

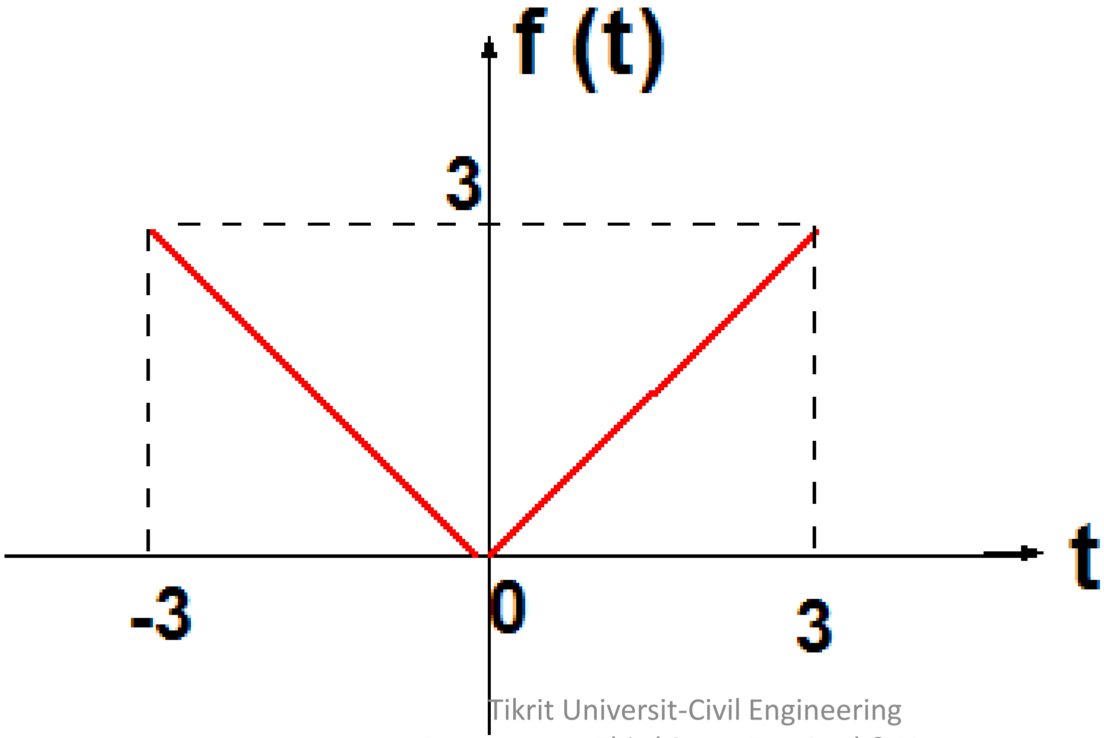


Example :

$$f(x)=\begin{cases}-t & -3<t<0\\ t & 0<t<3\end{cases}$$

Solution :

$$2p=6,\quad p=3\quad,\ take\ d=-3$$



$$a_o = \frac{1}{p} \int_{d}^{d+2p} f(t)\,dt$$

$$= \frac{1}{3}\left[\int_{-3}^{0} -t\,dt + \int_{0}^{3} t\,dt\right] = \frac{1}{3}\left(-\frac{t^2}{2}\Big|_{-3}^{0} + \frac{t^2}{2}\Big|_{0}^{3}\right) = 3$$

$\underline{\underline{OR:}}$

$$\frac{1}{3}(area) = \frac{1}{3}[2*(0.5*3*3)] = 3$$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\frac{n\pi t}{p}\,dt$$

$$= \frac{1}{3}\left[\int_{-3}^{0} -t\cos\frac{n\pi t}{3}\,dt + \int_{0}^{3} t\cos\frac{n\pi t}{3}\,dt\right]$$

$$= \frac{1}{3}\left[-\frac{3}{n\pi}t\sin\frac{n\pi t}{3} - \frac{9}{n^2\pi^2}\cos\frac{n\pi t}{3}\right]_{-3}^{0} +$$

$$\frac{1}{3}\left[\frac{3}{n\pi}t\sin\frac{n\pi t}{3} + \frac{9}{n^2\pi^2}\cos\frac{n\pi t}{3}\right]_{0}^{3}$$

$$= -\frac{1}{3}[0 + \frac{9}{n^2\pi^2} - (0 + \frac{9}{n^2\pi^2}\cos(-\pi n))] +$$

$$\frac{1}{3}[0 + \frac{9}{n^2\pi^2}\cos n\pi - (0 + \frac{9}{n^2\pi^2}\cos 0)]$$

$$= -\frac{3}{n^2\pi^2}(1 - \cos n\pi) + \frac{3}{n^2\pi^2}(\cos n\pi - 1) \qquad n \neq 0$$

$$\therefore a_n = \begin{cases} \frac{6}{n^2\pi^2}(\cos n\pi - 1) = \frac{-12}{n^2\pi^2} & (for\ \ n = odd) \\ 0 & (for\ \ n = even) \end{cases}$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\sin\frac{n\pi t}{p}\,dt$$

$$= \frac{1}{3}[\int_{-3}^{0} -t\sin\frac{n\pi t}{3}\,dt + \int_{0}^{3} t\sin\frac{n\pi t}{3}\,dt]$$

$$= -\frac{1}{3}\left[\frac{3}{n\pi}t\cos\frac{n\pi t}{3} + \frac{9}{n^2\pi^2}\sin\frac{n\pi t}{3}\right]_{-3}^{0} +$$

$$\frac{1}{3}\left[-\frac{3}{n\pi}t\cos\frac{n\pi t}{3} + \frac{9}{n^2\pi^2}\sin\frac{n\pi t}{3}\right]_{0}^{3}$$

$$= \frac{3}{n\pi}\cos(-n\pi) - \frac{3}{n\pi}\cos n\pi$$

$$\therefore b_n = 0$$

$$a_o = 3$$

$$a_n = \frac{-12}{\pi^2 n^2} \qquad (n = odd)$$

$$a_n = 0 \qquad (n = even)$$

$$b_n = 0$$

$$f(t) = \frac{a_o}{0} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi t}{p} + b_n \sin\frac{n\pi t}{p}\right]$$

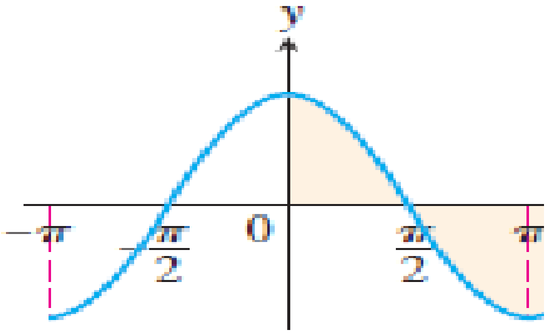
$$\therefore\ f(t) = \frac{3}{2} - \frac{12}{\pi}\left(\frac{1}{1}\cos\frac{\pi t}{3} + \frac{1}{9}\cos\frac{3\pi t}{3} + \frac{1}{25}\cos\frac{5\pi t}{3} + ...\right)$$

$\underline{Note:}$

$$f(t) \;\; is \;\; even \qquad \therefore b_n = 0$$

$$f(t) \;\; is \;\; odd \qquad \therefore a_n = 0$$

Even and Odd Functions :



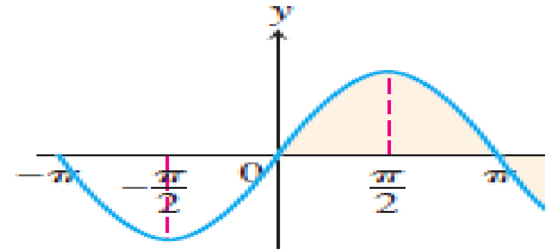
Even

Symmetrical with y-axis

g(-x)=g(x)

$$\int_{-p}^{p} g(x)dx = 2\int_{0}^{p} g(x)dx$$

$$\cos\frac{n\pi x}{p} \; is \; even$$



Odd

Symmetrical with Origin

g(-x)=-g(x)

$$\int_{-p}^{p} g(x)dx = 0$$

$$\sin\frac{n\pi x}{p} \; is \; odd$$

$\underline{Theorem\ (1):}$

$$If \quad f(x) \ \ is \ \ even, \ a_n = \frac{2}{p}\int_0^p f(t)\cos\frac{n\pi t}{p}\,dt$$

$$and \quad b_n = 0$$

$\underline{\underline{Theorem\ (2):}}$

$$If \quad f(x) \ \ is \ \ odd, \quad a_n = 0$$

$$and \quad b_n = \frac{2}{p}\int_0^p f(t)\sin\frac{n\pi t}{p}\,dt$$

$\underline{Product:}$

$$Odd * Odd = Even$$

$$Even * Even = Even$$

$$Even * Odd = Odd$$

$\underline{Half-range\ expansion:}$

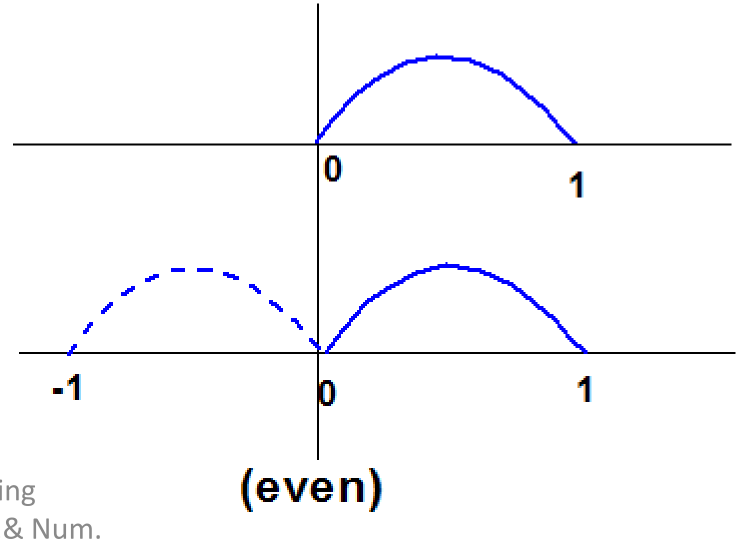
$Ex: Find\ the\ half-range\ of:$

$$f(t)=t-t^2 \qquad\qquad 0<t<1$$

$\underline{Solution:}$

$\underline{\underline{(1)}}$ $Extend\ \ to\ (-1,0)\ by$

$reflection\ \ in\ the\ \ y-axis$



$$b_n=0 \qquad\qquad (even)$$

$$a_n=\frac{2}{p}\int_0^p f(t)\cos\frac{n\pi t}{p}\,dt$$

$$a_n = \frac{2}{1}\int_0^1 (t - t^2)\cos\frac{n\pi t}{1}\,dt$$

$$= 2[(\frac{t}{n\pi}\sin n\pi t + \frac{1}{n^2\pi^2}\cos n\pi t) -$$

$$(\frac{t^2}{n\pi}\sin n\pi t + \frac{2t}{n^2\pi^2}\cos n\pi t - \frac{2}{n^3\pi^3}\sin n\pi t)]_0^1$$

$$= 2\left(\frac{\cos n\pi - 1}{n^2\pi^2} - \frac{2\cos n\pi}{n^2\pi^2}\right)$$

$$a_n = 2\left(\frac{\cos n\pi - 1}{n^2\pi^2} - \frac{2\cos n\pi}{n^2\pi^2} \right)$$

$$\therefore a_n = \frac{-2(1+\cos n\pi)}{n^2\pi^2} \qquad n \neq 0 \ \ (a_n = 0 \ \ n = odd)$$

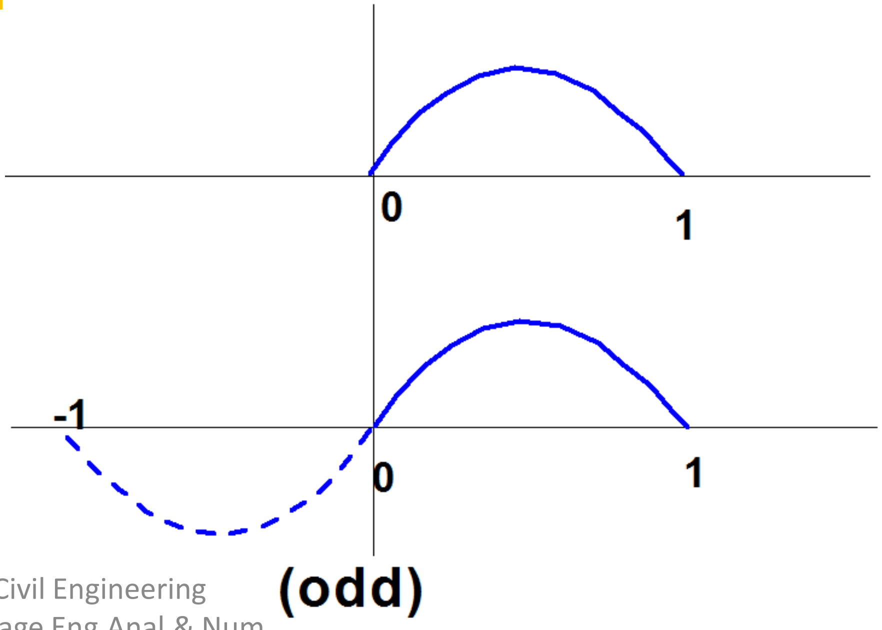
$$a_o = \frac{2}{1}\int_0^1 (t - t^2)dt = 2[\frac{t^2}{2} - \frac{t^3}{3}]_0^1 = \frac{1}{3}$$

$$\therefore \ f(t) = \frac{1}{6} - \frac{4}{\pi^2}[\frac{\cos 2\pi t}{4} + \frac{\cos 4\pi t}{16} + \frac{\cos 6\pi t}{36} + ...]$$

$\underline{\underline{(2)}}$ *Extend to* $(-1,0)$ *by reflection in the origin*

$$a_n = 0 \qquad\qquad (odd)$$

$$b_n = \frac{2}{p}\int_0^p f(t)\sin\frac{n\pi t}{p}\,dt$$



$$b_n = \frac{2}{1}\int_0^1 (t - t^2)\sin\frac{n\pi t}{1}\,dt$$

$$= 2[(\frac{-t}{n\pi}\cos n\pi t + \frac{1}{n^2\pi^2}\sin n\pi t) -$$

$$(\frac{-t^2}{n\pi}\cos n\pi t + \frac{2t}{n^2\pi^2}\sin n\pi t + \frac{2}{n^3\pi^3}\cos n\pi t)]_0^1$$

$$= 2[(\frac{-\cos n\pi}{n\pi}) - (\frac{-\cos n\pi}{n\pi} + \frac{2(\cos n\pi - 1)}{n^3\pi^3})]$$

$$b_n = 2[(\frac{-\cos n\pi}{n\pi}) - (\frac{-\cos n\pi}{n\pi} + \frac{2(\cos n\pi - 1)}{n^3 \pi^3})]$$

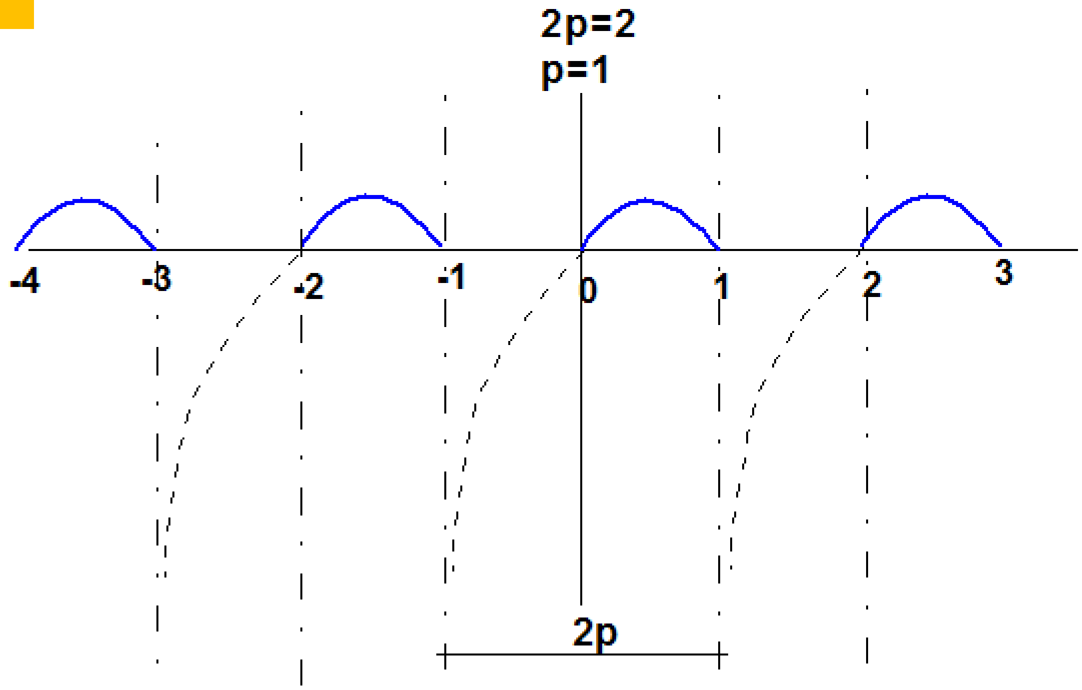
$$\therefore b_n = \frac{4(1-\cos n\pi)}{n^3 \pi^3} \qquad (b_n = 0, \quad n = even)$$

$$f(t) = \frac{8}{\pi^3}[\frac{\sin \pi t}{1} + \frac{\sin 3\pi t}{27} + \frac{\sin 5\pi t}{15} + \frac{\sin 7\pi t}{343} + ...]$$

(3) Extension of $(t-t^2)$ for $-1<t<0$

$$a_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\cos\frac{n\pi t}{p}\,dt$$

$$b_n = \frac{1}{p}\int_{d}^{d+2p} f(t)\sin\frac{n\pi t}{p}\,dt$$



$$a_n = \frac{1}{1}\int_{-1}^{1}(t-t^2)\cos\frac{n\pi t}{1}dt$$

$$=[(\frac{t}{n\pi}\sin n\pi t+\frac{1}{n^2\pi^2}\cos n\pi t)-$$

$$(\frac{t^2}{n\pi}\sin n\pi t+\frac{2t}{n^2\pi^2}\cos n\pi t-\frac{2}{n^3\pi^3}\sin n\pi t)]_{-1}^{1}$$

$$=\left(\frac{\cos n\pi t}{n^2\pi^2}-\frac{2\cos n\pi t}{n^2\pi^2}\right)_{-1}^{1}$$

$$a_n = \left(\frac{\cos n\pi t}{n^2\pi^2} - \frac{2\cos n\pi t}{n^2\pi^2} \right)_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}[(\cos n\pi - 2\cos n\pi) -$$

$$(\cos(-n\pi) - 2\cos(-n\pi))]$$

$$\underline{Note[\cos(-n\pi) = \cos n\pi]}$$

$$\therefore a_n = \frac{-4\cos n\pi}{n^2\pi^2} \qquad (n \neq 0)$$

$$a_o = \frac{1}{1}\int_{-1}^{1}(t - t^2)dt = -\frac{2}{3}$$

$$b_n = \frac{1}{1}\int_{-1}^{1}(t-t^2)\sin\frac{n\pi t}{1}\,dt$$

$$=[(\frac{-t}{n\pi}\cos n\pi t+\frac{1}{n^2\pi^2}\sin n\pi t)-$$

$$(\frac{-t^2}{n\pi}\cos n\pi t+\frac{2t}{n^2\pi^2}\sin n\pi t+\frac{2}{n^3\pi^3}\cos n\pi t)]_{-1}^{1}$$

$$=[(\frac{-t}{n\pi}\cos n\pi t)-(\frac{-t^2}{n\pi}\cos n\pi t+\frac{2}{n^3\pi^3}\cos n\pi t)]_{-1}^{1}$$

$$b_n = [(\frac{-\cos n\pi}{n\pi} + \frac{\cos n\pi}{n\pi} - \frac{2}{n^3\pi^3}\cos n\pi) -$$

$$(\frac{\cos(-n\pi)}{n\pi} + \frac{\cos(-n\pi)}{n\pi} - \frac{2}{n^3\pi^3}\cos(-n\pi))]$$

$$\therefore b_n = -2\frac{\cos n\pi}{n\pi}$$

$$\therefore a_n = \frac{-4\cos n\pi}{n^2\pi^2}$$

$$a_o = -\frac{2}{3}$$

$$b_n = \frac{-2\cos n\pi}{n\pi}$$

$$\therefore f(t) = -\frac{1}{3} + \frac{4}{\pi^4}\left(\frac{\cos \pi t}{1} - \frac{\cos 2\pi t}{4} + \frac{\cos 3\pi t}{9} - \frac{\cos 4\pi t}{16} + \frac{\cos 5\pi t}{25} \ldots\ldots\right) + \frac{2}{\pi}\left(\frac{\sin \pi t}{1} - \frac{\sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} - \frac{\sin 4\pi t}{4} \ldots\ldots\ldots\ldots\right)$$

Alternative forms of fourier Series:

The complex exponetial form is obtained:

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(a_n\frac{e^{\frac{ni\pi t}{p}}+e^{-\frac{ni\pi t}{p}}}{2}+b_n\frac{e^{\frac{ni\pi t}{p}}-e^{-\frac{ni\pi t}{p}}}{2i}\right)$$

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}e^{\frac{ni\pi t}{p}}+\frac{a_n+ib_n}{2}e^{-\frac{ni\pi t}{p}}\right)$$

Define:

$$C_o=\frac{a_o}{2};\quad C_n=\frac{a_n-ib_n}{2};\quad C_{-n}=\frac{a_n+ib_n}{2}$$

$$\therefore\ f(t)=\sum_{-\infty}^{\infty}C_n\,e^{\frac{ni\pi t}{p}}$$

$$\left\langle f(t)=\sum_{-\infty}^{\infty} C_n\ e^{\frac{n i \pi t}{p}} \right\rangle$$

$$C_o=\frac{1}{2}a_o=\frac{1}{2p}\int_{d}^{d+2p} f(t)\,dt$$

$$C_n=\frac{a_n-i b_n}{2}$$

$$=\frac{1}{2}[\frac{1}{p}\int_{d}^{d+2p} f(t)\cos\frac{n\pi t}{p}dt-i\frac{1}{p}\int_{d}^{d+2p} f(t)\sin\frac{n\pi t}{p}dt]$$

$$=\frac{1}{2p}\int_{d}^{d+2p} f(t)(\cos\frac{n\pi t}{p}-i\sin\frac{n\pi t}{p})dt$$

$$\therefore C_n=\frac{1}{2p}\int_{d}^{d+2p} f(t)\ e^{\frac{-n i \pi t}{p}}\ dt$$

$$C_{-n}=\frac{a_n+i b_n}{2}=\frac{1}{2p}\int_{d}^{d+2p} f(t)\ e^{\frac{n i \pi t}{p}}\ dt$$

$$\left\langle f(t)=\sum_{-\infty}^{\infty} C_n\, e^{\frac{n i \pi t}{p}} \right\rangle$$

$$C_o=\frac{1}{2}a_o=\frac{1}{2p}\int_{d}^{d+2p} f(t)\,dt$$

$$C_n=\frac{1}{2p}\int_{d}^{d+2p} f(t)\;\; e^{\frac{-n i \pi t}{p}}\; dt$$

$$C_{-n}=\frac{a_n+i\,b_n}{2}=\frac{1}{2p}\int_{d}^{d+2p} f(t)\;\; e^{\frac{n i \pi t}{p}}\; dt$$

$\underline{\underline{OR}}:$

$$\left\langle C_n=\frac{1}{2p}\int_{d}^{d+2p} f(t)\;\; e^{\frac{-n i \pi t}{p}}\; dt \right\rangle \quad (n=+ve,-ve\; or\; 0)$$

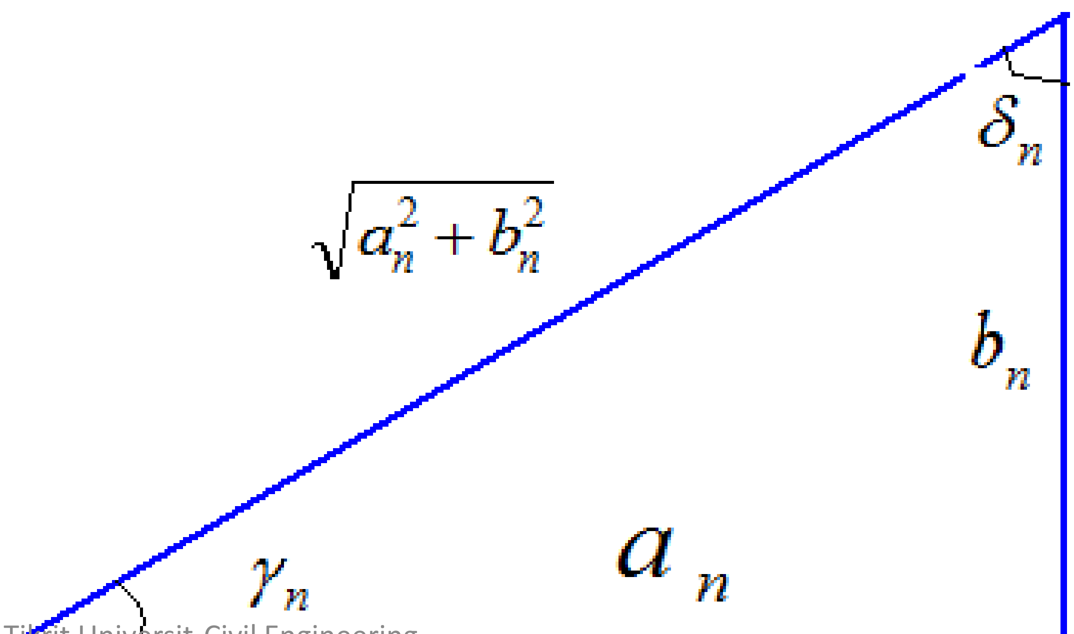
Another trigonometric forms of fourier Series :

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{p}+b_n\sin\frac{n\pi t}{p}\right)$$

$$f(t)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\sqrt{a_n^2+b_n^2}\left(\frac{a_n}{\sqrt{a_n^2+b_n^2}}\cos\frac{n\pi t}{p}+\frac{b_n}{\sqrt{a_n^2+b_n^2}}\sin\frac{n\pi t}{p}\right)$$

$$Let\ A_o=\frac{a_o}{2}$$

$$A_n=\sqrt{a_n^2+b_n^2}$$



$$f(t) = A_o + \sum_{n=1}^{\infty} A_n \left(\cos\frac{n\pi t}{p} \cos\gamma_n + \sin\frac{n\pi t}{p} \sin\gamma_n \right)$$

$$= A_o + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{p} - \gamma_n\right)$$

$\underline{\underline{OR:}}$

$$f(t) = A_o + \sum_{n=1}^{\infty} A_n \left(\cos\frac{n\pi t}{p} \sin\delta_n + \sin\frac{n\pi t}{p} \cos\delta_n \right)$$

$$= A_o + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi t}{p} + \delta_n\right)$$

$\underline{\underline{Note}}:$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

By solving the above equations :

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$\underline{Example}:$ $Find\ \ the\ complex\ \ form\ of\ Fourier$

$$Series\ of:\ f(t)=e^{-t} \qquad -1<t<1$$

$\underline{Solution:}$

$$C_n=\frac{1}{2p}\int_{d}^{d+2p} f(t)e^{-\frac{ni\pi t}{p}}\,dt=\frac{1}{2}\int_{-1}^{1}e^{-t}\ e^{-ni\pi t}dt$$

$$=\frac{1}{2}\int_{-1}^{1}e^{-(1+ni\pi)t}\ dt \quad =\frac{1}{2}\left[\frac{e^{-(1+ni\pi)t}}{-(1+ni\pi)}\right|_{-1}^{1}$$

$$C_n = \frac{1}{2}\left[\frac{e^{-(1+ni\pi)t}}{-(1+ni\pi)}\right|_{-1}^{1}$$

$$= \frac{e^{-(1+ni\pi)} - e^{(1+ni\pi)}}{-2(1+ni\pi)} = \frac{e.e^{ni\pi} - e^{-1}.e^{-ni\pi}}{2(1+ni\pi)}$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$e^{-i\pi} = \cos\pi - i\sin\pi = -1$$

$$e^{ni\pi} = e^{-ni\pi} = (-1)^n$$

$$C_n = \frac{e.e^{ni\pi} - e^{-1}.e^{-ni\pi}}{2(1+ni\pi)} = \frac{(-1)^n}{1+ni\pi}\frac{e-e^{-1}}{2}$$

$$= \frac{(-1)^n(1-ni\pi)\sinh 1}{1+n^2\ \pi^2}$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{ni\pi t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-ni\pi)\sinh 1}{1+n^2\ \pi^2} e^{ni\pi t}$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \& \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$This\ can\ be\ converted\ \text{into}\ trigonometric\ series:$$

$$C_n = \frac{a_n - i\,b_n}{2} \qquad \& \qquad C_{-n} = \frac{a_n + i\,b_n}{2}$$

$$Solve\ the\ above\ Eqs.$$

$$a_n = C_n + C_{-n} \qquad \& \qquad b_n = i\,(C_n + C_{-n})$$

$$a_n = \frac{(-1)^n (1 - n\,i\,\pi)\sinh 1}{1 + n^2\ \pi^2} + \frac{(-1)^{-n} (1 + n\,i\,\pi)\sinh 1}{1 + n^2\ \pi^2}$$

$$= \frac{(-1)^n (2\sinh 1)}{1 + n^2\ \pi^2}$$

$$b_n = i\left[\frac{(-1)^n (1 - n\,i\,\pi)\sinh 1}{1 + n^2\ \pi^2} - \frac{(-1)^{-n} (1 + n\,i\,\pi)\sinh 1}{1 + n^2\ \pi^2}\right]$$

$$= \frac{(-1)^n\, 2\,n\,\pi \sinh 1}{1 + n^2\ \pi^2}$$

$$a_n = \frac{(-1)^n(2\sinh 1)}{1+n^2\ \pi^2}$$

$$b_n = \frac{(-1)^n 2n\ \pi \sinh 1}{1+n^2\ \pi^2}$$

$$\therefore\ f(t) = \sinh 1 - 2\sinh 1\left(\frac{\cos \pi t}{1+\pi^2} - \frac{\cos 2\pi t}{1+4\pi^2} + \frac{\cos 3\pi t}{1+9\pi^2}\right.$$

$$\left. - \frac{\cos 4\pi t}{1+16\pi^2} + ...\right) - 2\pi \sinh 1\left(\frac{\sin \pi t}{1+\pi^2} - \frac{2\sin 2\pi t}{1+4\pi^2} + \right.$$

$$\left.\frac{3\sin 3\pi t}{1+9\pi^2} - \frac{4\sin 4\pi t}{1+16\pi^2} + ...\right)$$

$\underline{Note:}(-1)^0 = 1 \quad \& \quad (-1)^n = (-1)^{-n}$

Sum of functions :

Fourier coefficients of Sum.

$$f_1 + f_2$$

are Sum. of f_1 & f_2

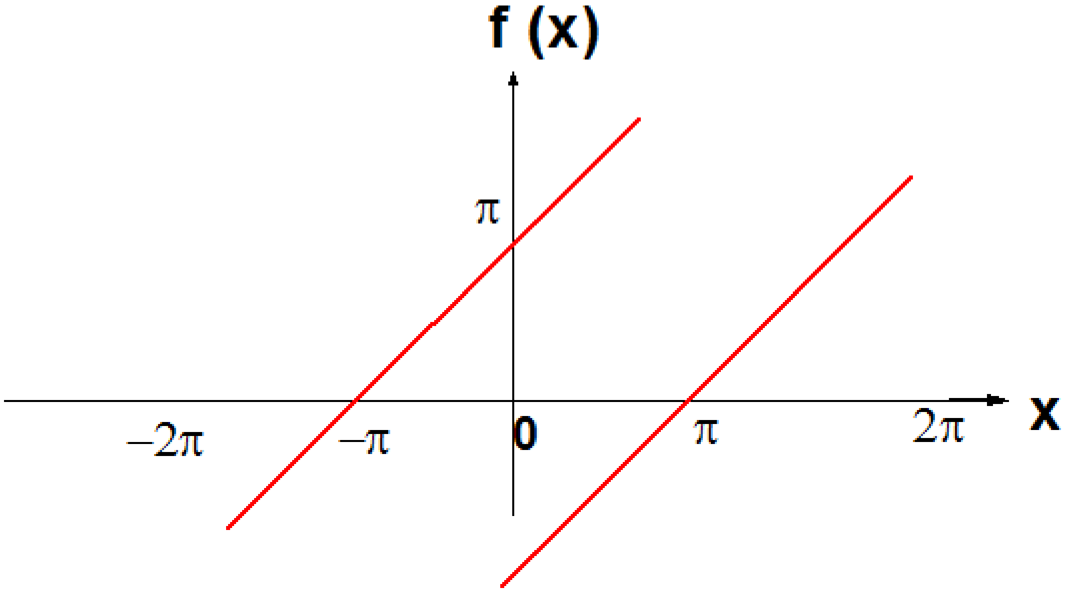
Example :

$$f(x) = x + \pi \qquad -\pi < x < \pi$$

$$f = f_1 + f_2 \quad \text{where} \quad f_1 = x \quad \& \quad f_2 = \pi$$

Solution :

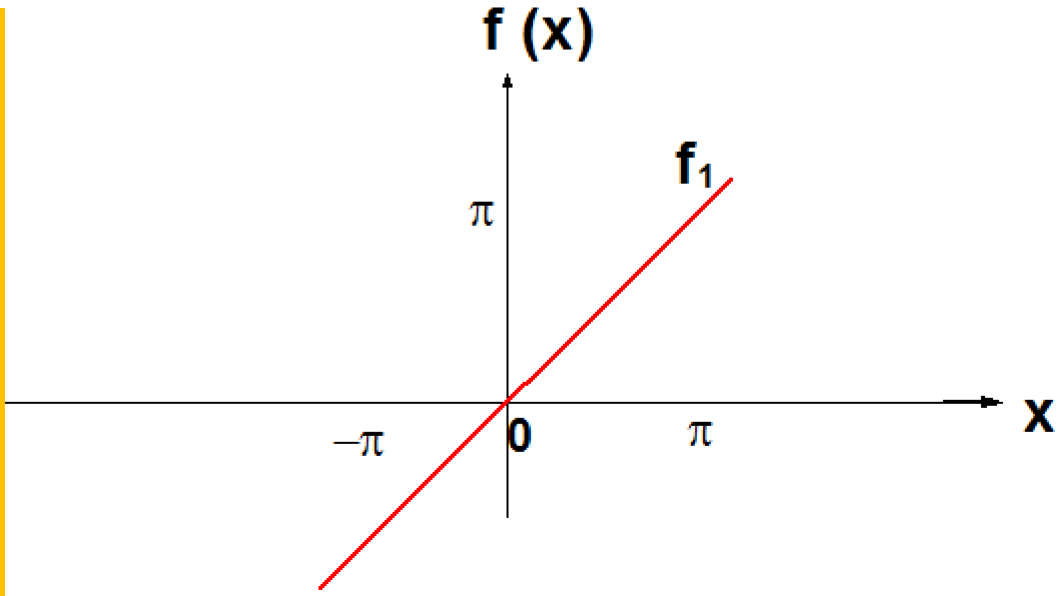
The coefficients of Fourier series of f_2 *are zero except for* $a_o = 2\pi$



$$Since \quad f_1 \ is \ odd, \quad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\, \sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, \sin nx\, dx$$



$$b_n = \frac{2}{\pi}[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\Big|_0^{\pi}$$

$$= -\frac{2}{\pi}[+\frac{\pi}{n}\cos n\pi - 0] = -\frac{2}{n}\cos n\pi$$

$$f_2 \Rightarrow a_o = 2\pi$$

$$f_1 \Rightarrow a_n = 0 \quad \& \; b_n = -\frac{2}{n}\cos n\pi$$

$$(b_1 = 2,\; b_2 = \frac{-2}{2},\; b_3 = \frac{2}{3},\; b_4 = \frac{-2}{4},.....)$$

$$\therefore\; f(x) = \pi + 2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + ...$$

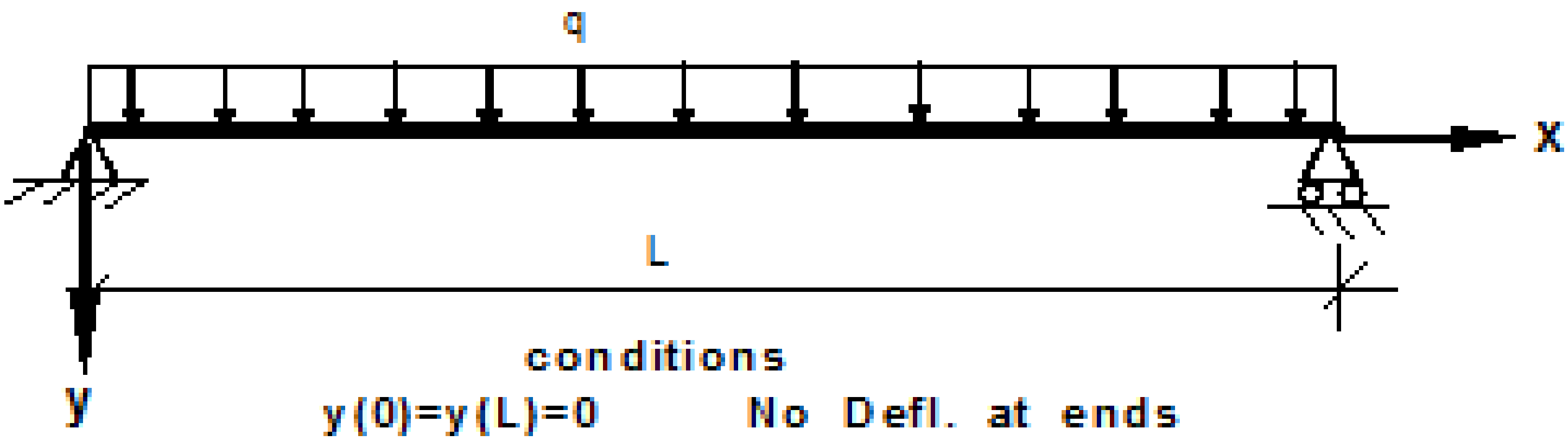
$$= \pi + 2(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + ...)$$

$$\underline{\underline{Applications:}}$$

$$\underline{*\ Deflection\ of\ simply\ supported\ beams:}$$

$\underline{Note:}$

$$EIy'' = M\ \ (moment),\ \ EIy''' = V\ \ (shear)\ \&\ EIy^{iv} = W\ \ (load)$$



conditions

y(0)=y(L)=0 No Defl. at ends

y"(0)=y"(L)=0 No Mom. at ends

$$EI\frac{d^4y}{dx^4}=q(x)$$

$$\frac{d^4y}{dx^4}=\frac{q(x)}{EI} \qquad (1)$$

$y=y(x)$, *to satisfy the boundary conditions at the ends*; $x=0$ & $x=L$

$$Try\ y=\sum_{m=1}^{\infty}C_m\sin\frac{m\pi x}{L} \qquad (2)$$

$y=0$, *where* $\sin 0=\sin m\pi=0$

$$Expand\ \ the\ load\ \ by\ \frac{1}{2} range\ (\sin e\, series)$$

$$q(x) = \sum b_m \sin \frac{m\pi x}{L} \qquad\qquad (3)$$

$$b_m = \frac{2}{L}\int_0^L q(x)\,\sin\frac{m\pi x}{L}\,dx$$

$$Subst.\ Eqs.\ (2)\ \&\ (3)\ \mathrm{int}\,o\ \ Eq.\ (1):$$

$$\sum C_m \left(\frac{m\pi}{L}\right)^4 \sin\frac{m\pi x}{L} = \frac{1}{EI}\sum b_m \sin\frac{m\pi x}{L}$$

$$C_m \left(\frac{m\pi}{L}\right)^4 = \frac{b_m}{EI}$$

$$\therefore C_m = \frac{b_m}{EI}\left(\frac{L}{m\pi}\right)^4$$

For Uniform Distributed Load :

$$b_m = \frac{2}{L}\int_0^L q \sin\frac{m\pi x}{L}\,dx$$

$$= -\frac{2q}{L}\,\frac{L}{m\pi}\cos\frac{m\pi x}{L}\Big|_0^L$$

$$= -\frac{2q}{m\pi}[\cos m\pi - \cos 0]$$

$$= -\frac{2q}{m\pi}[(-1)^m - 1]$$

$$\therefore b_m = \begin{cases} \dfrac{4q}{m\pi} & m = odd \\ 0 & m = even \end{cases}$$

$$\therefore C_m = \frac{4q}{EI\, m\pi}\left(\frac{L}{m\pi}\right)^4 = \frac{4qL^4}{EI\, m^5 \pi^5}$$

$$y = \sum C_m \sin\frac{m\pi x}{L}$$

$$= \sum_{m=1,3,5,..}^{\infty} \frac{4qL^4}{EI\, m^5 \pi^5} \sin\frac{m\pi x}{L}$$

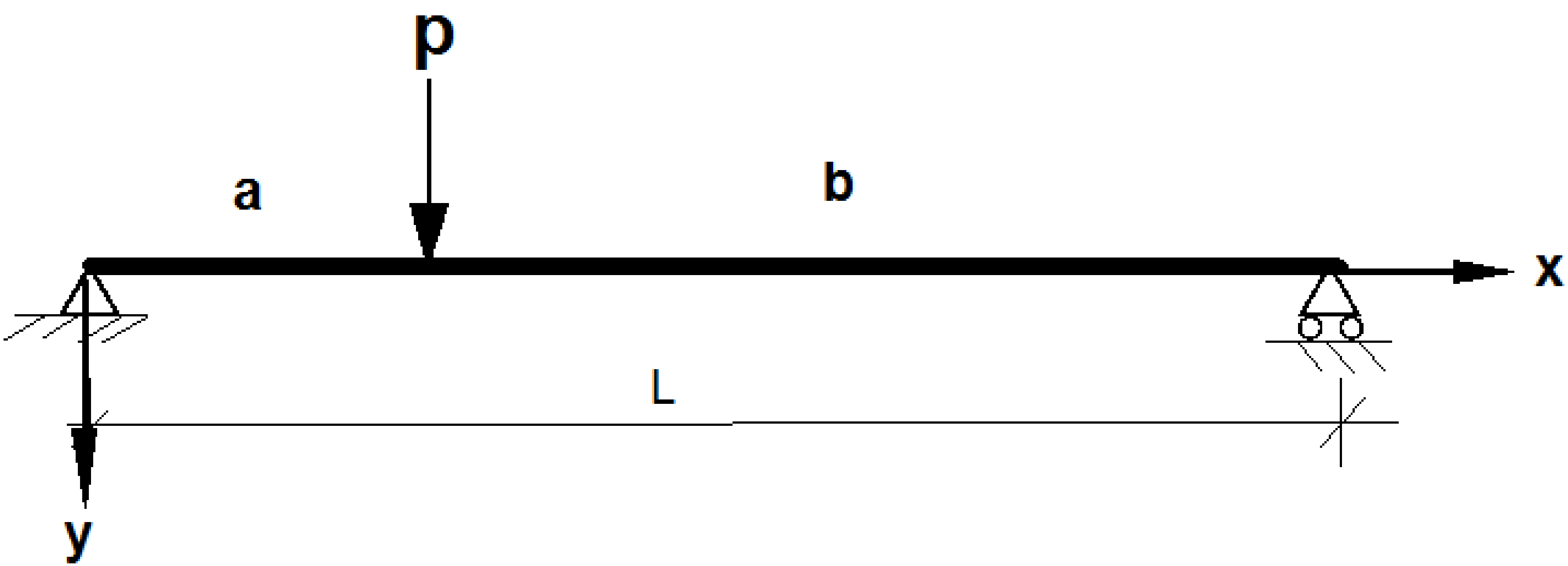
$$Take \quad m = 1 \quad at \quad x = \frac{L}{2}$$

$$y = \frac{4qL^4}{EI\ \pi^5} \sin\frac{\pi\, L/2}{L} = \frac{4qL^4}{EI\ \pi^5} \sin\frac{\pi}{2}$$

$$y = 0.013071054 \ \frac{qL^4}{EI}$$

$$y_{exact} = \frac{5qL^4}{384EI} = 0.01302833 \ \frac{qL^4}{EI}$$

$$\underline{Example}: Simply \; \sup ported \; beam,$$
$$with \; concentrated \; load.$$



From previous example:

$$\frac{d^4 y}{dx^4} = \frac{q(x)}{EI} \qquad (1)$$

$$y = \sum_{m=1}^{\infty} C_m \sin\frac{m\pi x}{L} \qquad (2)$$

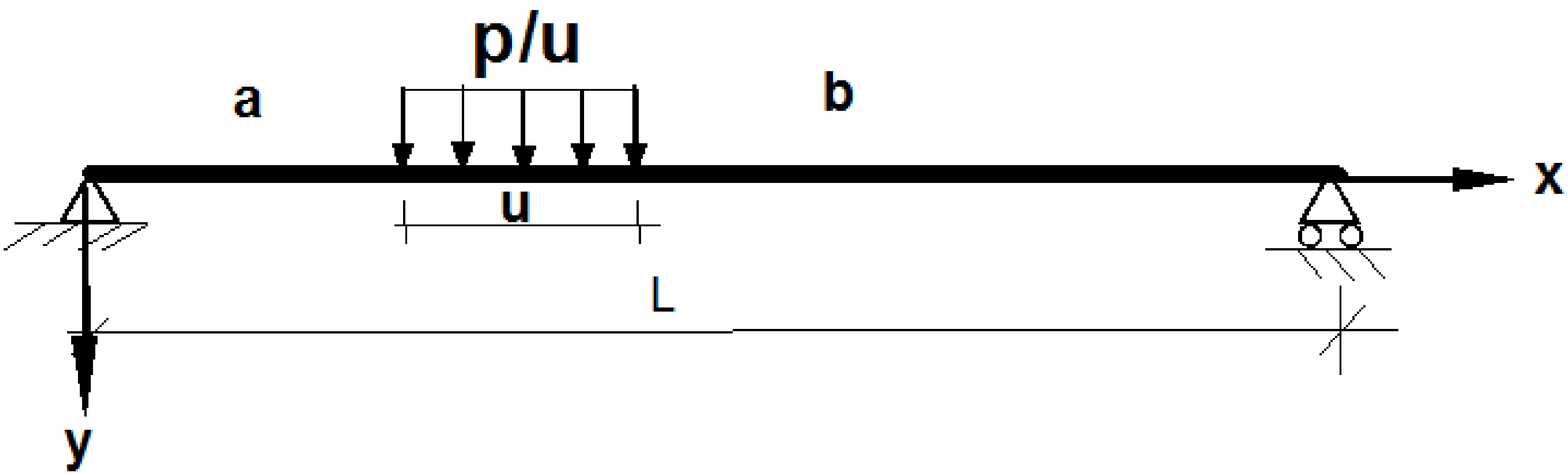
$$q(x) = \sum b_m \sin\frac{m\pi x}{L} \qquad (3)$$

$$b_m = \frac{2}{L}\int_0^L q(x)\sin\frac{m\pi x}{L}\,dx$$

$$b_m = \frac{2}{L}\int_a^a p\sin\frac{m\pi x}{L}\,dx = 0\ ?????????$$

$$Distribute\ \ p\ over\ small\ \ dis\tan ce\ \ u:$$

$$b_m = \frac{2}{L}\int_a^{a+u} \frac{p}{u} \sin\frac{m\pi x}{L} dx$$



$$b_m = -\frac{2}{L}\frac{p}{u}\frac{L}{m\pi}\cos\frac{m\pi x}{L}\Big|_a^{a+u}$$

$$b_m = -\frac{2p}{um\pi}\left[\cos\frac{m\pi(a+u)}{L} - \cos\frac{m\pi a}{L}\right]$$

$$= -\frac{2p}{um\pi}\left[\cos\frac{m\pi a}{L}\cos\frac{m\pi u}{L} - \right.$$

$$\left.\sin\frac{m\pi u}{L}\sin\frac{m\pi a}{L} - \cos\frac{m\pi a}{L}\right]$$

$$u \to 0 \quad \therefore \cos\frac{m\pi u}{L} \to 1$$

$$\& \ \sin\frac{m\pi u}{L} \to \frac{m\pi u}{L}$$

$$b_m = -\frac{2p}{um\pi}[\cos\frac{m\pi a}{L} - \frac{m\pi u}{L}\sin\frac{m\pi a}{L} - \cos\frac{m\pi a}{L}]$$

$$= -\frac{2p}{um\pi}[-\frac{m\pi u}{L}\sin\frac{m\pi a}{L}]$$

$$\therefore b_m == -\frac{2p}{L}\sin\frac{m\pi a}{L}$$

$$C_m(\frac{m\pi}{L})^4 = \frac{b_m}{EI} \Rightarrow C_m = \frac{b_m}{EI}(\frac{L}{m\pi})^4$$

$$C_m = \frac{2p\sin\frac{m\pi a}{L}}{LEI}\frac{L^4}{m^4\pi^4}$$

$$\therefore C_m = \frac{2pL^3}{E\,\mathrm{Im}^4\pi^4}\sin\frac{m\pi a}{L}$$

$$C_m = \frac{2pL^3}{E\,\mathrm{Im}^4\pi^4}\sin\frac{m\pi a}{L}$$

$$y = \sum \frac{2pL^3}{E\,\mathrm{Im}^4\pi^4}\sin\frac{m\pi a}{L}\sin\frac{m\pi x}{L}$$

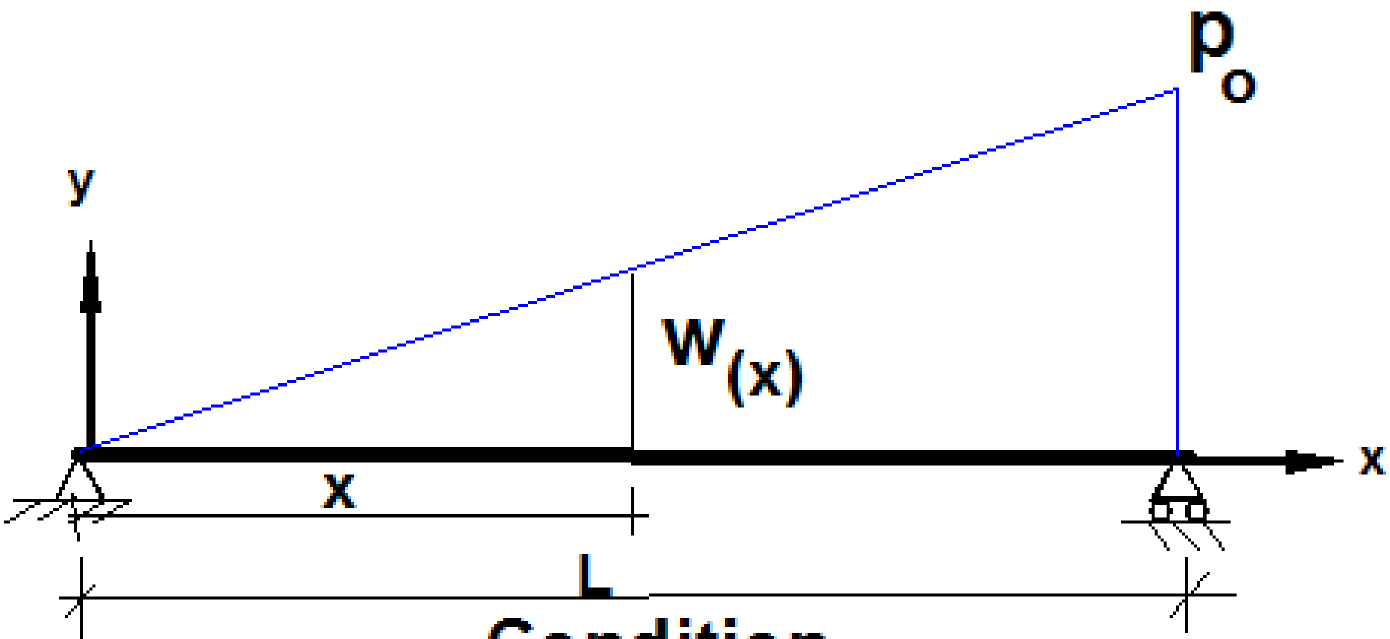
$$For\quad a = \frac{L}{2},\quad x = \frac{L}{2}\quad \&\quad m = 1$$

$$y = \frac{2pL^3}{EI\pi^4}\sin\frac{\pi}{2}\sin\frac{\pi}{2}$$

$$\therefore y = \frac{2pL^3}{EI\pi^4} = 0.0205319\frac{pL^3}{EI}$$

$$y_{exact} = 0.0208333\frac{pL^3}{EI}$$

$$\underline{Example}: Load\ w_{(x)} = \frac{p_o}{L} x \text{ and neglected the Wt. of the beam. Find the deflection curve.}$$



Condition

y(0) = y(L) = 0

y"(0)=y"(L)=0

$\underline{Solution :}$

$$EIy'' = M$$

$$EIy''' = V$$

$$EIy^{iv} = -W_{(x)} = -p_o \frac{x}{L} \qquad (1)$$

$We\ choose\ a \sin e\ series\ \ for\ \ y\ that\ \ satisfies\ \ Eq.(1)$

$$y = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \qquad (2)$$

$Because\ each\ term\ (and\ hence\ the\ Sum.\ itself\ )$

$satisfied\ the\ end\ conditions.\ Hence\ \exp and\ the$

$load\ deflection\ \ (-p_o \frac{x}{L})\, in\ a\ Fourier\ \sin e\ series$

$on\ \text{int}\, erval\ (0, L):$

$$b_n = \frac{2}{L}\int_0^L \left(\frac{-p_o x}{L}\right)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2p_o}{L^2}\left[x\frac{L}{n\pi}\cos\frac{n\pi x}{L} - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{2p_o}{L^2}\left[\left(\frac{L^2}{n\pi}\cos\frac{n\pi L}{L} - 0\right) - \frac{L^2}{n^2\pi^2}(\sin n\pi - \sin 0)\right]$$

$$\therefore b_n = \frac{2p_o}{L^2}\frac{L^2}{n\pi}\cos n\pi = \frac{2p_o}{n\pi}(-1)^n$$

$$\therefore -p_o \frac{x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$= \frac{2p_o}{\pi} \sum \frac{(-1)^n}{n} \sin \frac{n\pi x}{L}$$

Now compute y^{iv} *from* $Eq.(2)$ *and put* $\text{int}\, o$ $Eq.(1)$:

$$EI \frac{\pi^4}{L^4} \sum n^4 C_n \sin \frac{n\pi x}{L} = \frac{2p_o}{\pi} \sum \frac{(-1)^n}{n} \sin \frac{n\pi x}{L}$$

The coefficien ts of like term must be equal :

$$EI \frac{\pi^4}{L^4} n^4 C_n = \frac{2p_o}{\pi} \frac{(-1)^n}{n}$$

$$\therefore C_n = (-1)^n \frac{2p_o L^4}{EI \pi^5 n^5}$$

$$y=\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}$$

$$=\sum(-1)^n \frac{2p_o L^4}{EI\,\pi^5 n^5}\sin\frac{n\pi x}{L}$$

$$n=1$$

$$y=-\frac{2p_o L^4}{EI\,\pi^5}\sin\frac{\pi x}{L}$$

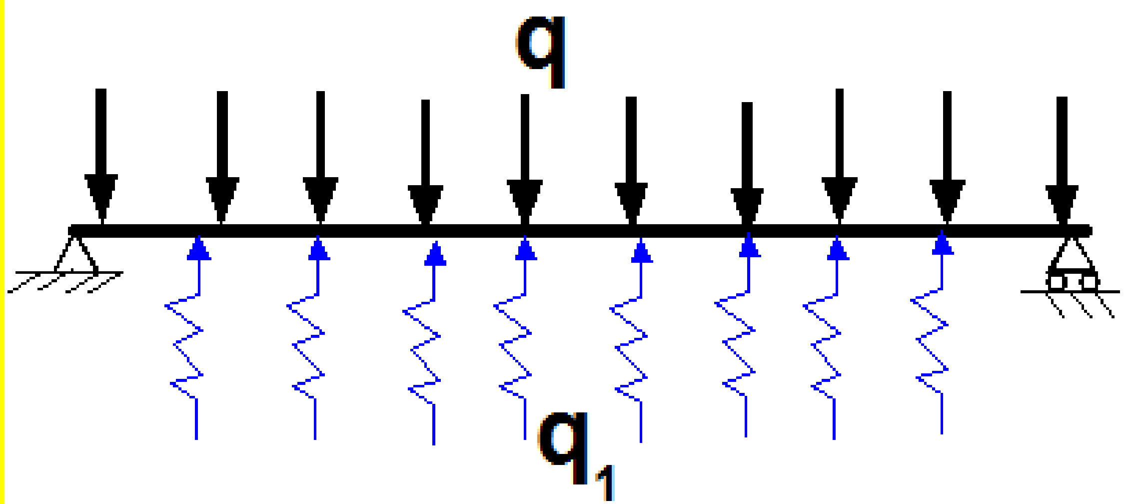
$$If\ \ x=\frac{L}{2}$$

$$y=-\frac{2p_o L^4}{EI\,\pi^5}\sin\frac{\pi}{2}=-0.00654\frac{p_o L^4}{EI}$$

* Beam on elastic foundation :

$$q_1 = k\,y$$

$$q_{(x)} = q - k\,y$$



$$\frac{d^4 y}{d x^4} = \frac{q - k\,y}{EI}$$

$$\frac{d^4 y}{d x^4} + \frac{k\,y}{EI} = \frac{q}{EI} \qquad (1)$$

$$Try \quad y = \sum C_m \sin \frac{m\pi x}{L}$$

$$q_{(x)} = \sum b_m \sin \frac{m\pi x}{L}$$

$$Subst.\ the\ above\ \operatorname{int} o\ \ Eq.\,(1):$$

$$\sum C_m \left(\frac{m\pi}{L}\right)^4 \sin \frac{m\pi x}{L} + \frac{k}{EI} \sum C_m \sin \frac{m\pi x}{L} = \frac{1}{EI} \sum b_m \sin \frac{m\pi x}{L}$$

$$C_m \left(\frac{m\pi}{L}\right)^4 + \frac{k}{EI} C_m = \frac{b_m}{EI}$$

$$C_m \left[\left(\frac{m\pi}{L}\right)^4 + \frac{k}{EI}\right] = \frac{b_m}{EI}$$

$$\therefore C_m = \frac{b_m}{EI\left[\left(\frac{m\pi}{L}\right)^4 + \frac{k}{EI}\right]}$$

For simply supported beam under uniform dead load on elastic foundation:

$$b_m = \frac{4q}{m\pi} \quad m = 1,3,5,\ldots..$$

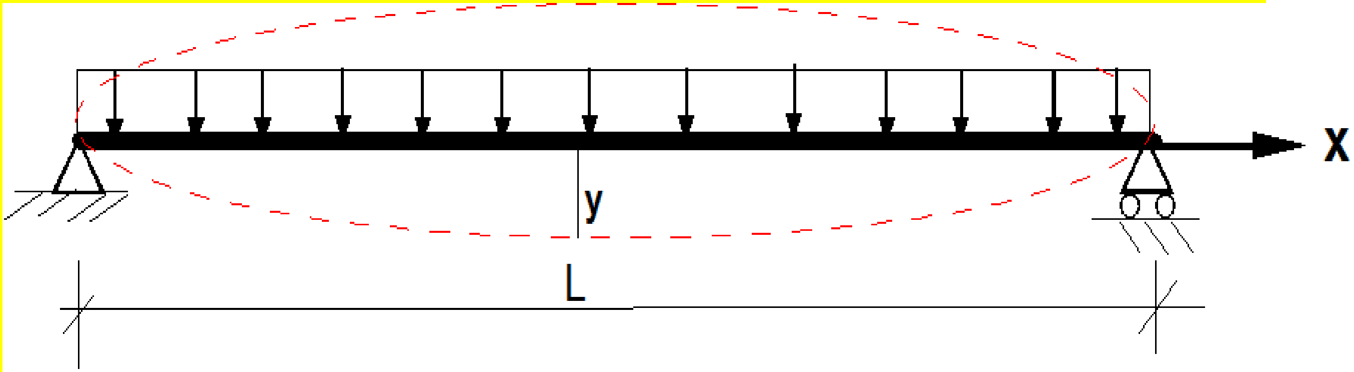
$$\therefore C_m = \frac{4q}{m\pi EI[(\frac{m\pi}{L})^4 + \frac{k}{EI}]}$$

$$y = \sum \frac{4q \sin \frac{m\pi x}{L}}{m\pi EI[(\frac{m\pi}{L})^4 + \frac{k}{EI}]}$$

* Free vibration of simply supported beams:

Free vibration is started after the load is removed

$$\frac{d^4y}{dx^4}=\frac{q}{EI}$$



$$Inertia\ \ force=-\frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2}\left(\begin{array}{c}\rho = Wt.\ \ per\ unit\ volume\\ A = cross\ section\ area\end{array}\right)$$

$$\frac{d^4y}{dx^4}+\frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2}=0 \qquad (1)$$

$$Try\ \ y(x,t) = \sum_{n=1}^{\infty} f(t) \sin \frac{m\pi x}{L} \qquad (2)$$

$$Subst.\ \ Eq.(2)\ \ \text{into}\ \ Eq.(1):$$

$$\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^4 f(t) \sin \frac{n\pi x}{L} + \frac{\rho A}{EI} \sum f''(t) \sin \frac{n\pi x}{L} = 0$$

$$\left(\frac{n\pi}{L}\right)^4 f(t) + \frac{\rho A}{EI} f''(t) = 0$$

$$\underline{\underline{OR}}:$$

$$\frac{\rho A}{EI} f''(t) + \left(\frac{n\pi}{L}\right)^4 f(t) = 0$$

$$f''(t) + \frac{EI}{\rho A} \left(\frac{n\pi}{L}\right)^4 f(t) = 0$$

$$f''(t)+\frac{EI}{\rho A}\left(\frac{n\pi}{L}\right)^4 f(t)=0$$

$$Let\ \frac{EI}{\rho A}\left(\frac{n\pi}{L}\right)^4=\alpha^2$$

$$\therefore\ f''(t)+\alpha^2 f(t)=0$$

$$Let\ f(t)=c\,e^{mt}$$

$$\therefore m^2+\alpha^2=0 \Rightarrow m=\pm\alpha\, i$$

$$f(t)=c_1\cos\ \alpha t+c_2\sin\ \alpha t$$

$$\therefore\ y(x,t)=\sum(c_1\cos\ \alpha t+c_2\sin\ \alpha t)\sin\frac{n\pi x}{L}$$

$$At\ \ t=0 \Rightarrow velocity=0=\frac{\partial y}{\partial t}$$

$$\frac{\partial y}{\partial t}=\sum(-c_1\alpha\sin\ \ \alpha t+c_2\,\alpha\cos\,\alpha\,t)\sin\frac{n\pi x}{L}$$

$$\frac{\partial y}{\partial t}=0\ \ @\ t=0$$

$$0=\sum(-c_1\alpha\sin\,0+c_2\,\alpha\cos\,0)\sin\frac{n\pi x}{L}$$

$$0=\sum(0+c_2\,\alpha)\Rightarrow c_2=0$$

$$y = \sum c_1 \cos \alpha t \sin \frac{n\pi x}{L}$$

$$At\ t = 0, \quad y(x,0) = y\ (static)$$

$$For\ uniform\ distribution\ load:$$

$$y = \sum \frac{4q\ L^4}{EI\ n^5\ \pi^5} \sin \frac{n\pi x}{L}$$

$$At\ t = 0, \quad y(x,0) = y\ (static):$$

$$\sum \frac{4q\ L^4}{EI\ n^5\ \pi^5} \sin \frac{n\pi x}{L} = \sum c_1 \cos 0 \sin \frac{n\pi x}{L}$$

$$\therefore c_1 = \frac{4qL^4}{EI\ n^5\ \pi^5}$$

$$\therefore y(x,t) = \sum \frac{4qL^4}{EI\ n^5 \pi^5} \cos \alpha t \sin \frac{n\pi x}{L}$$

The Fourier Integral:

Many problems do not involve periodic function, and an non periodic functions cannot be handled directly by Fourier series. If in a periodic function:, $f_p(t)$, we let approaches infinity, then $f(t)$ is no longer periodic.

Begin with complex form:

$$f_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{ni\pi t}{p}}$$

$$C_n = \frac{1}{2p}\int_{-p}^{p} f_p(t) e^{-\frac{ni\pi t}{p}} dt$$

$$= \frac{1}{2p}\int_{-p}^{p} f_p(\tau) e^{-\frac{ni\pi \tau}{p}} d\tau$$

$$\therefore f_p(t) = \sum_{n=-\infty}^{\infty} [\frac{1}{2p}\int_{-p}^{p} f_p(\tau) e^{-\frac{ni\pi \tau}{p}} d\tau] e^{\frac{ni\pi t}{p}}$$

$$= \sum_{n=-\infty}^{\infty} [\frac{1}{2\pi}\int_{-p}^{p} f_p(\tau) e^{-\frac{ni\pi \tau}{p}} d\tau] e^{\frac{ni\pi t}{p}} \frac{\pi}{p}$$

$$Let\ \ frequency\ \ w_n = \frac{n\pi}{p}\ and\ \ \Delta w = \frac{\pi}{p}$$

$$f_p(t) = \sum_{n=-\infty}^{\infty} [\frac{1}{2\pi} e^{i w_n t} \int_{-p}^{p} f_p(\tau) e^{-i w_n \tau} d\tau] \Delta w$$

$$Let\ F(w) = \frac{e^{i w_n t}}{2\pi} \int_{-p}^{p} f_p(\tau) e^{-i w_n \tau} d\tau$$

$$\therefore\ f_p(t) = \sum_{n=-\infty}^{\infty} F(w_n) \Delta w \qquad (1)$$

$$The\ \lim it\ \ of\ \ a\ \ sum\ of\ \ the\ \ form\,(1)\ is\ the$$

$$\text{int}\,egral,\ \int_{-\infty}^{\infty} F(w)\ dw$$

$Since\ \ p \to \propto\ implies\ \ \Delta w \to 0,\ the\ non\ periodic$

$\lim it\ of\ \ f_p(t),\ say\ \ f(t)\ can\ be\ written\ \ as\ the$

$\int egral;$

$$f(t) = \int_{-\propto}^{\propto} [\frac{1}{2\pi} e^{iwt} \int_{-\propto}^{\propto} f(\tau) e^{-iw\tau} d\tau] dw \qquad (2)$$

$The\ \ other\ \ \ from:$

$$f(t) = \int_{-\propto}^{\propto} g(w) e^{-iwt}\, dw \qquad (3)$$

$$g(w) = \frac{1}{2\pi} \int_{-\propto}^{\propto} f(\tau) e^{-iw\tau}\, d\tau \qquad (4)$$

$These\ two\ \ equations\ \ \ constitute\ \ what\ \ is\ \ known\ as$

$Fourier\ \ \ complex\ \ transform\ \ pair.$

$For\ trigonometric\ \ from:$

$Set\ (frequency);\ w_n = \dfrac{n\pi}{p},$

$$\Delta w_n = \frac{\pi(n+1)}{p} - \frac{n\pi}{p} = \frac{\pi}{p}\ \ or\ \ \underline{\underline{\frac{1}{p} = \frac{\Delta w}{\pi}}}$$

$$f_p(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty}[a_n \cos\frac{n\pi t}{p} + b_n \sin\frac{n\pi t}{p}]$$

$$= \frac{1}{2p}\int_{-p}^{p} f_p(\tau)\,d\tau + \frac{1}{\pi}\sum_{n=1}^{\infty}[\cos(wt)\,\Delta w\int_{-p}^{p} f_p(\tau)$$

$$\cos(w\tau)\,d\tau + \sin(wt)\,\Delta w\int_{-p}^{p} f_p(\tau)\sin(w\tau)\,d\tau]\ \ (5)$$

$$f_p(t)=\frac{1}{2p}\int_{-p}^{p} f_p(\tau)\,d\tau+\frac{1}{\pi}\sum_{n=1}^{\infty}[\cos(wt)\,\Delta w\int_{-p}^{p} f_p(\tau)$$

$$\cos(w\tau)\,d\tau+\sin(wt)\,\Delta w\int_{-p}^{p} f_p(\tau)\sin(w\tau)\,d\tau] \quad (5)$$

$Let\ \ p\rightarrow\infty(then\ \Delta w\rightarrow 0)\ and\ \ we\ can\ show;$

$$f(t)=\frac{1}{\pi}\int_{0}^{\infty}[\cos wt\int_{-\infty}^{\infty} f(\tau)\cos(w\tau)\,d\tau+\sin wt\int_{-\infty}^{\infty} f(\tau)\sin(w\tau)\,d\tau]dw$$

$\underline{\underline{OR}}:$

$$f(t)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(\tau)\cos w\tau \cos wt \, d\tau \, dw; \qquad (6)$$

$$f(t)\ \underline{\underline{is\ even}}[Fourier\ \text{cosine integral}]$$

$$f(t)=\frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(\tau)\sin w\tau \sin wt \, d\tau \, dw; \qquad (7)$$

$$f(t)\ \underline{\underline{is\ odd}}[Fourier\ \text{sine integral}]$$

It is convenient to have the Fourier cosine and sine integral representation as transform pair.

The Fourier cosine transform pair is:

$$\left.\begin{aligned} f(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(w) \cos wt \; dw \\ g(w) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos wt \; dt \end{aligned}\right\} \quad f(t) \text{ is even}$$

We have made it instead of τ.

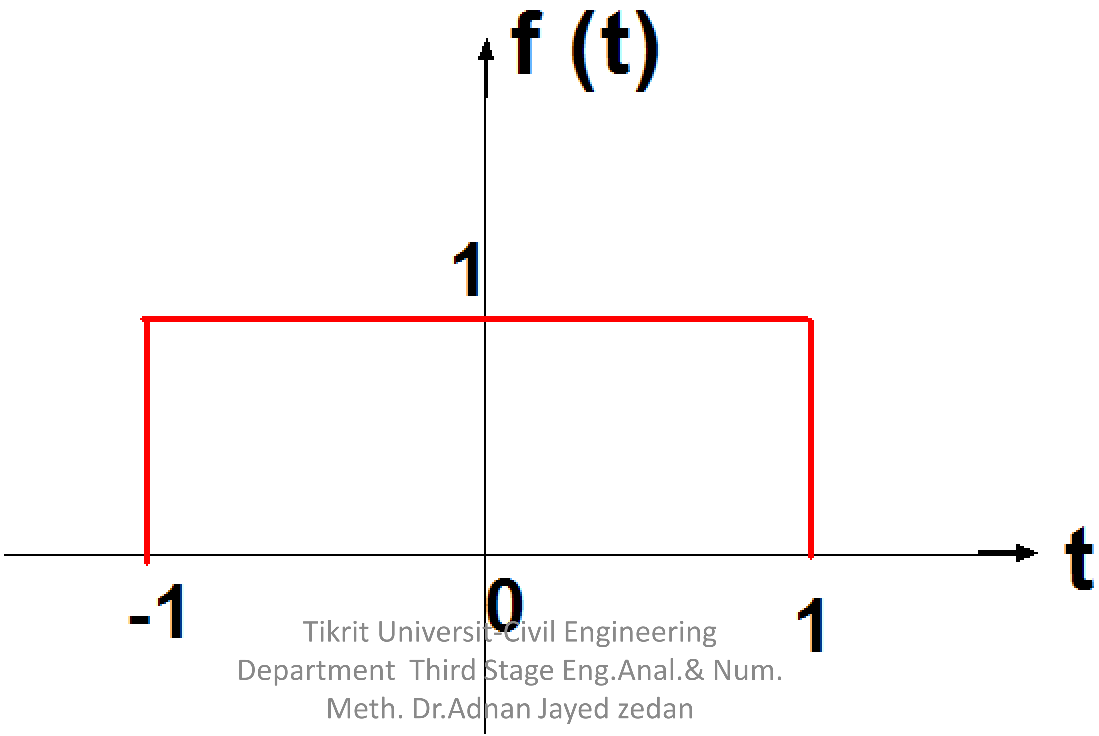
The Fourier sine transform pair is:

$$\left.\begin{aligned} f(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(w) \sin wt \; dw \\ g(w) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin wt \; dt \end{aligned}\right\} \quad f(t) \text{ is odd}$$

Example :

Find the Fourier integral of :

$$f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$



$\underline{Solution:}$

$The \quad function \quad is \;\; even; So:$

$$g(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos wt \; dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1} 1 \cos wt \; dt = \sqrt{\frac{2}{\pi}} \left. \frac{\sin wt}{w} \right|_0^1$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(w) \cos wt \; dw$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\sin w}{w} \cos wt \; dw \qquad (a)$$

The avergage of the left and right hand limit of :

$$f(t)@t=1 \text{ is equal to } \frac{|t_o|}{2}=\frac{1}{2}$$

$$\int_0^{\infty}\frac{\cos wt\sin w}{w}\,dw=\begin{cases}\frac{\pi}{2} & 0\le t\le 1\\ \frac{\pi}{4} & t=1\\ 0 & t>1\end{cases}$$

When $t=0$, *then* $\int_0^{\infty}\frac{\sin w}{w}\,dw=Si(\infty)=\frac{\pi}{2}.$

$The\ \ integral\ is\ the\ limit\ of\ the\ So-called\ ,$

$$\underline{\sin e\ \ integral}\ [Si(t) = \int_0^t \frac{\sin w}{w}\, dw \quad as\ \ t \to \infty.$$

$In\ \ non\ periodic\ case,\ as\ we\ did\ in\ the\ Fourier\ series,\ it\ is\ of\ interest\ to\ investigate\ how\ well\ the\ Fourier\ integral\ represents\ a\ function.$

$Let\ us\ return\ to\ our\ example;\ From\ Eq.(a)\ as$

$an\ approximation\ to\ f(t);$

$$we\ have\ the\ finite\ integral\ \frac{2}{\pi}\int_0^{w_o}\frac{\sin w}{w}\cos wt\ dw$$

$by\ replacing\ \infty\ by\ w_o.$

$$And\ since\ \cos a \sin b = \frac{\sin(a+b)-\sin(a-b)}{2}$$

$$\therefore \quad \frac{2}{\pi}\int_0^{w_o}\frac{\sin w}{w}\cos wt \; dw$$

$$=\frac{1}{\pi}\int_0^{w_o}\frac{\sin w(t+1)}{w}\,dw-\frac{1}{\pi}\int_0^{w_o}\frac{\sin w(t-1)}{w}\,dw$$

$$Set \;\; w+wt=u; \quad then \quad dw=\frac{du}{t+1}$$

$$\frac{d\,w}{w}=\frac{du}{w(t+1)}=\frac{du}{u} \;\; and \;\; \begin{cases} 0\le w\le w_o & approaches \\ 0\le u\le (t+1)w_o \end{cases}$$

$$Also\quad set\ w-wt=-u \Rightarrow wt-w=u$$

$$and\quad 0\le w\le w_o \qquad approaches \qquad 0\le u\le (t-1)w_o$$

<u>*Then we obtain:*</u>

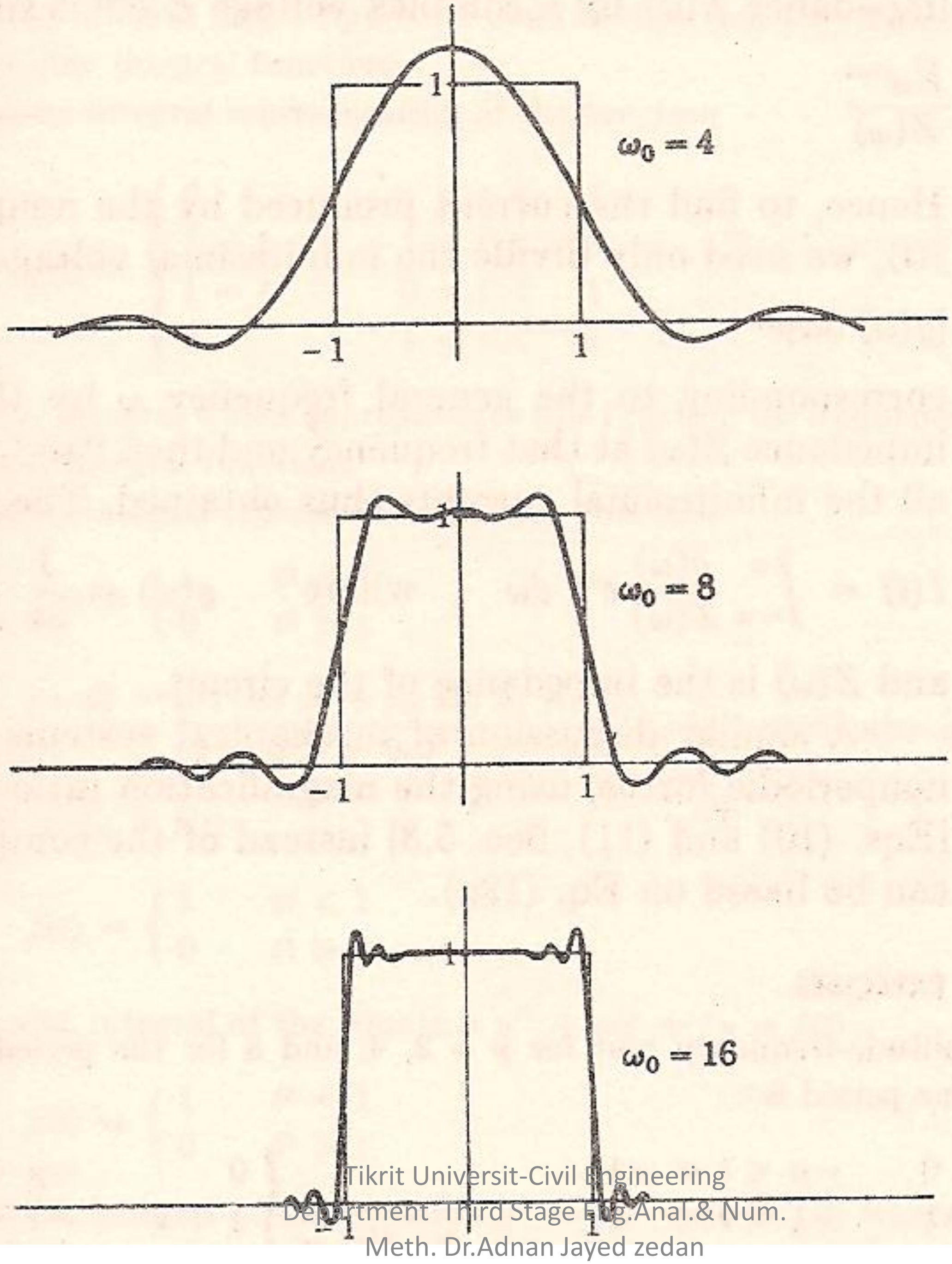
$$\frac{2}{\pi}\int_0^{w_o}\frac{\cos\ wt\sin\ w}{w}\,dw$$

$$=\frac{1}{\pi}\int_0^{w_o(t+1)}\frac{\sin u}{u}\,du-\frac{1}{\pi}\int_0^{w_o(t-1)}\frac{\sin u}{u}\,du$$

$$=\frac{1}{\pi}Si[w_o(t+1)]-\frac{1}{\pi}Si[w_o(t-1)]$$

The figure below shows this approximation for w_o=4, 8 and 16 rad/unit time.

Physically speaking, these curves describe the output of an ideal low-pass filter, cutting off all frequencies above w_o, when the input signal is an isolated rectangular pulse.


1
$\omega_0 = 4$
−1
1
1
$\omega_0 = 8$
−1
1
1
$\omega_0 = 16$
−1
1